A Factor Influencing Network Formation: Link Investment Substitutability

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Abstract

Existing literature on network formation usually proceeds under one of the following two assumptions regarding how links between players are built: a pair of players choose whether or not to have a link with each other and the link forms if both sides say, or at least one side says, yes. The assumptions suffer from two limitations. First, they overlook the tendency that individuals not only decide who to link with but also how much to invest in each relationship. Second, they imply either a complete equal or unequal devotion into a link when real-life experiences often lie somewhere in between. I propose a general approach to link formation by allowing weighted link investment and adopting a CES link formation function. The CES function bridges the two commonly employed link formation assumptions and parametrizes a novel feature in link formation: link investment substitutability. I apply this approach to two well-known network formation models: the connections model and the law of the few model, and find that link investment substitutability has a great impact on network formation. In the connections model, I find that perfect substitutability between link investments reduces the strong tension between efficiency and stability characterized before to zero. In the law of the few model, I show that greater link investment substitutability makes players specialize in networking or other activities.

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1 Introduction

The study of network formation has been an active research field for the past two decades. Economists are interested in this topic because we do observe individuals, firms, and nations actively managing their relationships, and the resulting network can have substantial social and economic implications.[1]

Theoretical investigations on the topic build upon assumptions on how two agents can form a link between them and on how a network affects the utility of players. Existing works, while making various modelling choices on network utility specifications, typically adopt the bilateral link formation assumption or the unilateral link formation assumption. The two assumptions both assume that agents make binary decisions on whether or not to invest in a link, and they differ in the requirement for the creation of a one: The bilateral assumption calls for mutual investments whereas the unilateral assumption demands effort from only one side. More specifically, let $a_{ij} \in \{0, 1\}$ be the choice i makes on whether to invest in a link with j and g_{ij} be the relationship status between i and j, the bilateral link formation assumption assumes that $g_{ij} = \min\{a_{ij}, a_{ji}\}$ and the unilateral link formation assumption assumes $g_{ij} = \max\{a_{ij}, a_{ji}\}$.

These two commonly used specifications are restrictive in two senses. First, they do not allow different intensities of link investments. Second, they essentially impose either an extremely reciprocal way of link formation or a completely unequal one. In practice, agents choose how much to spend on a relationship and links can be sponsored in flexible manners. For example, Kovanen et al. (2010) show with a mobile phone dataset that the communication network is weighted and that both sides of a link make positive effort (make calls) to maintain a contact but their devotion to the relationship (frequency of initiating a call) is different. Vaquera and Kao (2008) show with the Add Health data that friendships among adolescents have different intensities and can be unbalanced. We can also observe from our own research experience that we allocate time to projects differently and the contribution each author makes to a joint work are not necessarily of equal value.[2]

In this paper, we relax the above restrictions by allowing continuous link investment choice $a_{ij} \in \mathbb{R}_+$ and modelling link formation with the constant elasticity of substitution (CES) function:

$$g_{ij} = \begin{cases} h(a_{ij}, a_{ji}) = (\frac{1}{2}a_{ij}^{\beta} + \frac{1}{2}a_{ji}^{\beta})^{\frac{1}{\beta}} & \text{for weighted link formation} \\ \mathbb{1}\{h(a_{ij}, a_{ji}) \geq 1\} = \mathbb{1}\{(\frac{1}{2}a_{ij}^{\beta} + \frac{1}{2}a_{ji}^{\beta})^{\frac{1}{\beta}} \geq 1\} & \text{for unweighted link formation} \end{cases}$$

where $\mathbb{1}$ is the indicator function and β is a parameter that can take any real value. This specification permits a more flexible way for players to form links and allows us

[1] See Goyal (2007) and Jackson (2010) for a review.

[2] See the following papers for a further reference on the presence and implications of weighted and unequally sponsored links: Granovetter (1973), Newman (2001), Yook et al. (2001), Garlaschelli and Loffredo (2004), and Squartini et al. (2013).

to analyze equality of link investments, which is a feature of network structures that receives little attention from existing literature.

Our link formation assumption has two other related advantages.

First, the CES function nests the bilateral and unilateral link formation function as its special cases when β approaches negative and positive infinity respectively. This generalization enables us to analyze to what extent results from existing works are robust to small deviations from the link formation assumption they build upon. Moreover, not surprisingly, network formation models with the bilateral and the unilateral link formation assumption normally generate very different predictions on stable networks. The bridging of the two assumptions achieved with the CES specification facilitates investigations on how the characterizations change under one assumption to another, gradually or abruptly.

Second, the parameter β has an economic meaning. It captures the substitutability of link investments in forming a connection. For a greater β, there is a smaller drop in the marginal enhancement of $(\frac{1}{2}a_{ij}^{\beta} + \frac{1}{2}a_{ji}^{\beta})^{\frac{1}{\beta}}$ as a_{ij} rises alone, which indicates that it is less costly to have disproportionate a_{ij} and a_{ji}, i.e. we have a higher degree of link investment substitutability. So, by imposing different values of β, we can examine the role of this substitutability in network formation processes.

We believe that in different economic and social contexts, the degree of link investment substitutability is different. For example, in the case of communication networks, the communication technology and communication content both affect link investment substitutability. In terms of technology, the degree of substitutability rises as we move from face-to-face contacts to phone calls and then to email correspondences since demand less and less joint devotion. In terms of content, communication of knowledge that involves constant questioning and answering features lower link investment substitutability while communication of simple facts features higher link investment substitutability. In the case of trade networks, the kind of pre-investment needed for trade determines link investment substitutability. If the pre-investment is mostly in forms of learning regulations and customs of patners, then the degree of link investment substitutability is low. If the pre-investment is mostly physical, e.g. building a bridge or highway, then the degree of link investment substitutability is high. Finally, in the case of research network, link investment substitution in different subject can be different. Theoretical works that requires a lot of joint modelling and solving have low link investment substitutability whil empirical works where the jobs of modelling, data collection, and estimation are relatively more independent have higher link investment substitutability.

We apply the CES formulation to well-known models that lie in two broad classes of problems: pure network formation games and network formation with assorted activities games.[3] We select the *connections model* analyzed by Jackson and Wolinsky

[3] See Mauleon and Vannetelbosch (2016) for a recent survey on network formation games. Papers that study network formation with assorted activities include Baetz (2015), Hiller (2017), and König et al. (2014).

(1996) that features pure network formation and extend its bilateral link formation assumption with the unweighted CES link formation specification.[4] We pick the *law of the few* model studied by Galeotti and Goyal (2010) that models a simultaneous choice of network formation and another activity and replace its unilateral link formation assumption with the weighted CES link formation specification.

We now summarize the modelling and results from the two applications.

In the application to the connections model, each player is endowed with one unit of intrinsic value that can be shared through a network. Direct and indirect connections facilitate exchanges of the benefit, but there is a loss in exchange efficiency for any additional player that stands in between the exchange. Players form links in an unweighted CES manner with costly relationship investment to maximize the gathering of worth.

We first characterize the efficient network structure under the setup: Proposition 1 shows that it must be the complete network, the star, or the empty network depending on the value of β and other parameters of the model. When $\beta < 1$ so that link investments are strategic complements, the characterization is identical to that in Jackson and Wolinsky (1996). When $\beta > 1$ so that link investments are strategic substitutes, a greater β makes a denser network efficient in more situations. We then propose the solution concept of weighted pairwise stability as a generalization for the notion of pairwise stability. Proposition 2 illustrates that link investment substitutability plays a role in determining structures of stable networks. In particular, the star network is more likely to be stable with greater link investment substitutability. Finally, we look at the relationship between efficiency and stability by examining whether the efficient network is stable. Proposition 3 demonstrates that it is generally true that there exists situations where the efficient network is not stable. However, when $\beta = 1$ so that link investments are perfect substitutes, an efficient network is always stable. Moreover, as β gets closer to 1, the range of situations where efficiency and stability are in conflict shrinks.

The above results show that the efficient network characterization in Jackson and Wolinsky (1996) (which features the complete network, the star, and the empty network) can be generalized to setups with weighted link investment and CES link formation technology. More importantly, we point out that although the tension between efficiency and stability illustrated in their paper is common, it is much less prominent when we move away from the bilateral link formation assumption.

In the application to the law of the few model, players aim to collect information and there are two ways for them to do so: to search on their own or to invest in relationships that are created with the weighted CES technology and get a proportion

[4] The connections model is also discussed in Bala and Goyal (2000) as one of the cases they study. They analyze the setup with the unilateral link formation assumption which is a limiting case of our CES extension when $\beta \to +\infty$. Feri (2007), Hojman and Szeidl (2008) and Bloch and Dutta (2009) work on extensions of the connections model.

of information searched by those they relate to.[5] Players get utility from information and pay costs for searching and link investments.[6]

We provide the Nash equilibrium solution of the game in Proposition 4 and 5 for different ranges of β. We find that when β is smaller than a cutoff value that is between 0 and 1, all players search for some information and links are sponsored bilaterally. The equilibrium network structure can be combinations of isolated, regular and bipartite components which do not put connected players in very different network positions. When β is greater than 1, players specialize in searching or in building connections and all links are formed unilaterally. Equilibrium network structures, e.g. the star, feature distinct network position for connected players. When β is between the cutoff value and 1, equilibrium can be a mixture of the two forms summarized above.

The above characterization points to a general trend of moving towards a dispersion of searching, inequality in link investment, and a less regular network structure when β rises. Proposition 6 provides further support for this finding by showing that there is a smooth growth in cross-player searching and link investment difference as β increases.

Proposition 7 looks at the welfare distribution of players and shows that a larger β indicates an advantage of being a searcher because it makes it easier for the searcher to attract link investments from others.

Note that our characterization of equilibrium when $\beta > 1$ is analogous to that obtained in Galeotti and Goyal (2010), so it suggests that their results are robust to weighted link formation and a wider range of link formation technology. When link investments are more complementary ($\beta < 1$), we make predictions different from theirs.

This paper contributes to research in network formation. It bridges the two rather different bilateral and unilateral link formation assumptions that are commonly employed in existing literature and introduces the notion of link investment substitutability and shows its relevance in network formation. Olaizola and Valenciano (2015) also unify bilateral and unilateral link formation. They assume binary link investments and set the strength of a link as 0 when no sides makes investment, $\alpha \in [0, 1]$ when one side makes investment, and 1 when both sides make investments. The bilateral and the unilateral link investment assumptions are special cases of their specification when α takes the value of 0 and 1 respectively. Our apparoach can be considered as a generalization of their method for weighted link investments: By restricting attention to binary link investments and setting $\alpha = (\frac{1}{2})^{\frac{1}{\beta}}$ when $\beta > 0$ and $\alpha = 0$ when $\beta \leq 0$, our CES assumption is the same as their specification. Elliott (2015) studies the implication of link investment substitutability in trade networks, but his attension

[5] The object players try to obtain need not be restricted to information, it can be any public good that is shared with neighbours, e.g. influence.

[6] Kinateder and Merlino (2017) and Sethi and Yildiz (2016) study similar models where agents form networks and acquire information from neighbours. The local public good provision setup of the law of the few model was first analyzed by Bramoulle and Kranton (2007) in a fixed network.

is limited to the cases of perfect complements ($\beta \to -\infty$) and of perfect substitutes ($\beta = 1$).

Our work falls into the strand of literature on weighted network formation. Works in the field include Rogers (2008), Bloch and Dutta (2009), Baumann (2019), and Griffith (2019). Rogers (2008) analyzes a directed weighted network formation problem where agents allocate their link investment budget strategically to maximize their Bonacich centrality. This is quite different from our undirected network formation framework. Our paper is closer to the other three studie.

Bloch and Dutta (2009) analyze a weighted version of the connections model. They assume that the strength of a link is an additively separable and convex function of link investments. Our CES specification, when β takes the value of 1, satisfies their link formation assumption. The results we have for the connections model application when $\beta = 1$ also matches their characterization that the star network is both efficient and stable. Baumann (2019) looks at a weighted network formation model where players maximize their aggregate link strength including self-loops. She assumes a link formation function that features constant returns to scale, zero production with unilateral effort, and complementarity of link investments. Our CES specification, when $\beta \leq 0$ satisfies these conditions. She shows that in equilibrium, links can be sponsored in equal or unequal manners and the network can consist of regular and bipartite components. The law of the few application we analyze, although is with a different utility function, generates similar predictions when $\beta \leq 0$. We differ from Bloch and Dutta (2009) and Baumann (2019) by introducing variation in link formation technologies and investigating the impact of the varying variable.

Griffith (2019) studies a model similar to Baumann (2019) with not only constant but also decreasing and increasing returns to scale link formation technology. He shows that this difference in link formation has great implication on equilibrium networks. Our study focues on the implication of differenct link formation technologies along another dimension: link investment substitutability.

The rest of the paper is organized as follows. In the next section, we give a detailed discussion of the CES link formation assumption. We explain the model and results for the connections model and law of the few model with the CES specification in Section 3 and 4 respectively. Section 5 concludes. All proofs can be found in the Appendix.

2 The CES Link Formation Assumption

For a set of players $N = \{1, ..., n\}$, let $a_{ij} \in \mathbb{R}_+$ be the link investment from player i to j. We assume that a link g_{ij} between i and j is produced with inputs a_{ij} and a_{ji} and a symmetric CES link formation technology:

$$g_{ij} = \begin{cases} h(a_{ij}, a_{ji}) = (\frac{1}{2}a_{ij}^{\beta} + \frac{1}{2}a_{ji}^{\beta})^{\frac{1}{\beta}} & \text{for weighted link formation} \\ \mathbb{1}\{h(a_{ij}, a_{ji}) \geq 1\} = \mathbb{1}\{(\frac{1}{2}a_{ij}^{\beta} + \frac{1}{2}a_{ji}^{\beta})^{\frac{1}{\beta}} \geq 1\} & \text{for unweighted link formation} \end{cases}$$

where $\mathbb{1}$ is the indicator function and β is a parameter that can take any real value.

The specification can be employed to model both weighted and unweighted link formation. The weighted formulation assumes a CES production function directly. For the unweighted version, we assume that the sustainment of a link takes a level of overall effort aggregated by the CES function and two players can contribute in different ways to satisfy the requirement.

The CES function was first introduced in Arrow et al. (1961) as a way to generalize the Leontiff production function (of which the elasticity of substitution is 0) and the Cobb Douglas production function (of which the elasticity of substitution is 1) and is widely employed in macroeconomic literatures. The standard two factor CES function is:

$$\hat{h}(a_{ij}, a_{ji}) = \gamma(\alpha a_{ij}^{\beta} + (1-\alpha)a_{ji}^{\beta})^{\frac{1}{\beta}}$$

where $\gamma > 0$, $\alpha \in (0,1)$, and $\beta \leq 1$. Here we normalize γ to 1 since we will later introduce a link formation cost variable. We assume links are undirected, i.e. $\hat{h}(a_{ij}, a_{ji}) = \hat{h}(a_{ji}, a_{ij})$, which requires $\alpha = (1-\alpha) = 1/2$. So we arrive at:

$$h(a_{ij}, a_{ji}) = (\frac{1}{2}a_{ij}^{\beta} + \frac{1}{2}a_{ji}^{\beta})^{\frac{1}{\beta}}$$

that only has one parameter β, which we allow to take values greater than 1 to achieve the goal of including the unilateral link formation function as a special case of our formulation.

The CES assumption encompasses a wide range of specific link formation protocols. We have:

(i) $\lim_{\beta \to -\infty} h(a_{ij}, a_{ji}) = \min\{a_{ij}, a_{ji}\}$: the bilateral link formation assumption is a special case of our formulation.

(ii) $h(a_{ij}, a_{ji}) \mid_{\beta=1} = \frac{1}{2}(a_{ij} + a_{ji})$: this depicts the case where link investments are perfect substitutes and can be used to model scenarios where only the sum of investments matter for the formation of a link, e.g. rail and bridge construction. It can also be adopted to incorporate the possibility of side payments.

(iii) $\lim_{\beta \to 0} h(a_{ij}, a_{ji}) = \sqrt{a_{ij}a_{ji}}$: this Cobb-Douglas case can be a convenient functional form to work with when link investments are strategic complements.

(iv) $\lim_{\beta \to +\infty} h(a_{ij}, a_{ji}) = \max\{a_{ij}, a_{ji}\}$: the unilateral link formation assumption is a special case of our formulation.

The parameter β captures the substitutability of link investments in forming a connection. When $\beta < 1$, the second-order mixed derivative of h is positive:

$$\frac{\partial h(a_{ij}, a_{ji})}{\partial a_{ij} \partial a_{ji}} |_{\beta<1} > 0$$

so as one player increases his investments in a link, the other player will find it easier to create or strengthen their link with her investments. Link investments are strategic complements when $\beta < 1$.

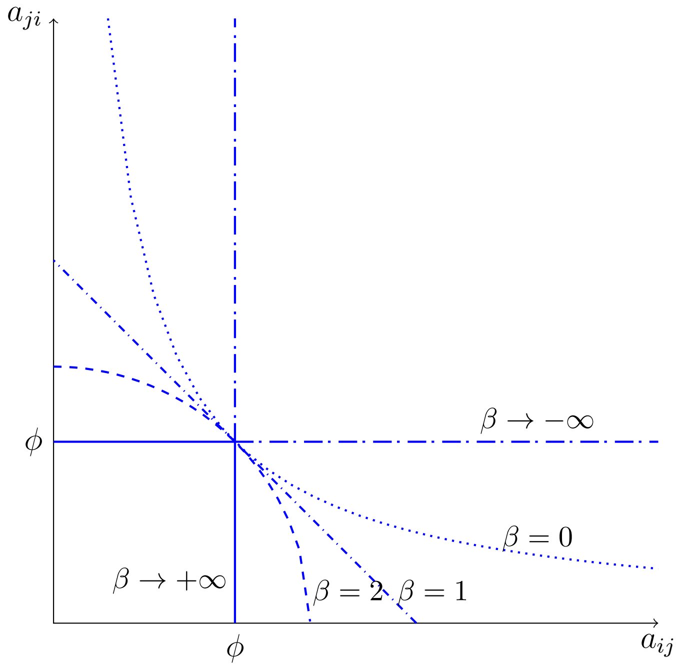


Figure 1: Isoquants for $h(a_{ij}, a_{ji}) = \phi$ under different values of β

When $\beta > 1$, the second-order mixed derivative of h is negative:

$$\frac{\partial h(a_{ij}, a_{ji})}{\partial a_{ij} \partial a_{ji}}|_{\beta>1} < 0$$

so as one player increases his investments in a link, the link investments from another player has less effect on the creation or strengthening of their link. Link investments are strategic substitutes when $\beta > 1$.

More specifically, as illustrated in Arrow et al. (1961), $\frac{1}{1-\beta}$ is the elasticity of substitution between a_{ij} and a_{ji}, so a greater β indicates a greater link investment substitutability.

Figure 1 pictures the isoquants of h for five different values of β and provides a graphical illustration of how β is related to link investment substitutability.

3 Application I: The Connections Model

The connections model was first studied in the seminal work of Jackson and Wolinsky (1996). In that paper, they introduce the solution concept of pairwise stability, characterize the efficient networks that maximize the aggregate utility of players and the pairwise stable networks respectively, and then examine the relationship between efficient and stable networks. The connections model is one of the two stylized models the above analysis was carried out upon. They show that there is a strong tension between efficiency and stability.

Their findings are based on the bilateral link formation assumption. In this section, we extend their analysis by allowing weighted link investments and adopting a more flexible CES link formation specification. We propose and employ the solution concept of weighted pairwise stability and show that the efficiency-stability tension is alleviated as link investments become perfect substitutes.

3.1 The Model

There is a set of players $N = \{1, ..., n\}$. Each player $i \in N$ chooses how much link investment to make to other players, which can be summarized by a vector $a_i = \{a_{i1}, ..., a_{in}\}$ where $a_{ij} \in A_{ij}$. The investment decisions of all agents $a = \{a_{ij}\}_{i,j \in N}$ is an investment profile, and we denote the space of investment profiles with $A = \prod_{i,j \in N} A_{ij}$.

An unweighted and undirected network $g = \{g_{ij}\}_{i,j \in N}$, where $g_{ij} = g_{ji} \in \{0, 1\}$ for all $i, j \in N$, specifies whether there is a link between any two players. We use $G = \{g \mid g_{ij} = g_{ji} \in \{0, 1\}\}$ to denote the set of networks n players can have.

Any investment profile creates an unweighted and undirected network. We use $g(a)$ to denote the network created with investment profile a.

The utility of a player is determined by the investment profile (and the network created with it). Before specifying the form of the utility function $(u_i(a))_{i \in N}$, we first define the efficiency and stability concept we adopt.

A network is *efficient* if it is created with an investment profile that maximizes the aggregate utility of players:

Definition 1. A network $g \in G$ is efficient if $g = g(a^*)$ where

$$a^* = \arg\max_{a \in A} \sum_{i \in N} u_i(a)$$

We propose the solution concept of *weighted pairwise stability* which adapts the notion of pairwise stability for weighted link investments. A network is weighted pairwise stable if it can be created with an investment profile where (i) no player wants to adjust his investment to any link, and (ii) no pair of unlinked players can find a way to form a mutually beneficial link. Formally, let $a_{-ij,kl,...}$ denote an investment profile a excluding the investment choices from i to j, k to l,...,

Definition 2. A network $g \in G$ is weighted pariwise stable if there exists an investment profile $a \in A$ such that $g = g(a)$ and for all pairs of players $(i, j) \in N^2$:

(i) if $g_{ij} = 1$, then for all $a'_{ij} \in A_{ij}$ and $a'_{ji} \in A_{ji}$:

$$u_i(a) \geq u_i(a'_{ij}, a_{-ij})$$
$$u_j(a) \geq u_j(a'_{ji}, a_{-ji})$$

(ii) if $g_{ij} = 0$, then there does not exist $a'_{ij} \in A_{ij}$ and $a'_{ji} \in A_{ji}$ such that

$$u_i(a'_{ij}, a'_{ji}, a_{-ij,ji}) \geq u_i(a)$$
$$u_j(a'_{ij}, a'_{ji}, a_{-ij,ji}) \geq u_j(a)$$

with at least one inequality being strict.

We now illustrate the relationship between the notions of weighted pairwise stability and pairwise stability.

The solution concept of pairwise stability can be employed for network formation models with the bilateral link formation assumption. In our framework, a network formation model with the bilateral link formation assumption is one where link investment choices are binary: $\forall i, j \in N : A_{ij} = \{0, 1\}$, a link is created with mutual investments: $\forall i, j \in N, a \in A : g_{ij}(a) = \min\{a_{ij}, a_{ji}\}$, and the utility of a player is decreasing with unproductive investments of his own and unaffected by unproductive investments of others: $\forall a, a' \in A$ such that $g(a) = g(a')$: $u_i(a) < u_i(a')$ if $a_i < a'_i$ and $u_i(a) = u_i(a')$ if $a_i = a'_i$.[7] Note that sinec a network $g \in G$ and an investment profile $a \in A$ are both $n \times n$ matrices and $G \subset A$, so $(u_i(g))_{i \in N}$ is well defined. We use $g - ij$ denote the network with the deletion of a link between i and j from g and $g + ij$ denote the network with the addition of a link between i and j to g.

Definition 3 (Jackson and Wolinsky (1996))**.** A network $g \in G$ is pariwise stable if for all pairs of players $(i, j) \in N^2$:

(i) if $g_{ij} = 1$, then

$$u_i(g) \geq u_i(g - ij)$$
$$u_j(g) \geq u_j(g - ji)$$

(ii) if $g_{ij} = 0$, then it cannot be

$$u_i(g + ij) \geq u_i(g)$$
$$u_j(g + ij) \geq u_j(g)$$

with at least one inequality being strict.

We show that for a network formation model with the bilateral link formation assumption, the notions of weighted pairwise stability and pairwise stability are equivalent.

Remark 1. *Consider a network formation model with the bilateral link formation assumption. A network $g \in G$ is weighted pairwise stable if and only if it is pairwise stable.*

[7] A natural form of utility function $u_i(a) = v_i(g(a)) - c_i(a_i)$ where c_i is an increasing function satisfies the last condition of bilateral link formation. The condition is also implied in Jackson and Wolinsky (1996) since they do not allow players to pay for anything other than existing links of theirs.

This result shows that the notion of weighted pairwise stability generalizes that of pairwise stability for richer network formation set-ups.

We now specify the assumptions for the *weighted connections model* we examine in this section. For all $i, j \in N$, we set $A_{ij} = \mathbb{R}_+$ so that players can choose any non-negative level of link investments. We assume links to be formed according to the unweighted CES link formation speicification

$$g_{ij}(a) = \begin{cases} 0 & \text{when } i = j \\ \mathbb{1}\{h(a_{ij}, a_{ji}) \geq 1\} = \mathbb{1}\{(\frac{1}{2}a_{ij}^{\beta} + \frac{1}{2}a_{ji}^{\beta})^{\frac{1}{\beta}} \geq 1\} & \text{otherwise} \end{cases}$$

and the utility of players to be

$$u_i(a) = \sum_{j \in N} \delta^{d_{ij}(g(a))} - c \sum_{j \in N} a_{ij}$$

where $\delta \in (0, 1)$ is a discount factor, $c > 0$ is the cost of link investments, and $d_{ij}(g)$ is the distance between i and j in network g. To be specific, the distance between i and j in g is the number of links in the shortest path between i and j in g. A path between i and j in g is a sequence of distinct players $i = v_0, v_1, ..., v_m = j$ such that $g_{v_{k-1}v_k} = 1$ for all $k = 1, ..., m$. We set $d_{ii}(g) = 0$ for all $i \in N$ and $d_{ij}(g) = \infty$ when there is no path between i and j under g.

The rationale behind the utility function is the following. We assume that each player possesses a unit of non-rivalrous intrinsic value that is shared through the network. There is a decay in the sharing efficiency if two players are far away from each other. This decay is captured by parameter δ. The utility a network provides to a player is the sum of value he obtains, and he pays a constant cost c for each unit of link investments he makes.

In our analysis, we will encounter three specific networks: the *complete network*, the *star network* and the *empty network*. The complete network is a network where all players are linked to each other: $\forall i \in N, j \neq i : g_{ij} = 1$. The star network is a network where all links are between one centre player and the rest of the players whom we call periphery players: $\exists m \in N$ such that $g_{mj} = 1$ for all $j \neq m$ and $g_{ij} = 0$ for all $i, j \neq m$. The empty network is a network with no links: $\forall i, j \in N : g_{ij} = 0$.

3.2 Results

We first solve for efficient networks.

Proposition 1. *Consider the weighted connections model. The efficient network is:*

(i) the complete network when $0 \leq c \leq \frac{1}{2}^{\frac{1}{\max\{\beta,1\}}-1}(\delta - \delta^2)$,

(ii) the star network when $\frac{1}{2}^{\frac{1}{\max\{\beta,1\}}-1}(\delta - \delta^2) \leq c \leq \frac{1}{2}^{\frac{1}{\max\{\beta,1\}}-1}(\delta + \frac{n-2}{2}\delta^2)$, *and*

(iii) the empty network when $\frac{1}{2}^{\frac{1}{\max\{\beta,1\}}-1}(\delta + \frac{n-2}{2}\delta^2) \leq c$.

The above efficient network characterization is similar to the one obtained in Jackson and Wolinsky (1996). When $\beta < 1$ so that link investments are strategic complements, the term $\frac{1}{2}^{\frac{1}{\max\{\beta,1\}}-1}$ equals to 1 and the two characterizations are identical. This is because when link investments are strategic complements, the most cost-effective way for two players to sponsor a link would be for them to both make one unit of investment to the other, making the total expenditure for a link $2c$, which is the same as the cost of a link assumed in Jackson and Wolinsky (1996).[8]

When $\beta > 1$ so that link investments are strategic substitutes, the value of β plays a role in the efficient characterization. Now, the most cost-effective way for two players to sponsor a link would be for one of them to invest all it takes to build a link while the other does not invest, making the total expenditure for a link $2^{\frac{1}{\beta}}c$ which drops in β. A greater β reduces the expenditure because it favors unilateral sponsorship more as can be seen in Figure 1. A denser network is hence more likely to be efficient under large β. Figure 2 depicts this effect: We can see that β is greater than 1 and rises, the complete network is preferred to the star network for a greater range of link investment cost c and the star network is preferred over the empty network for a greater range of cost c.

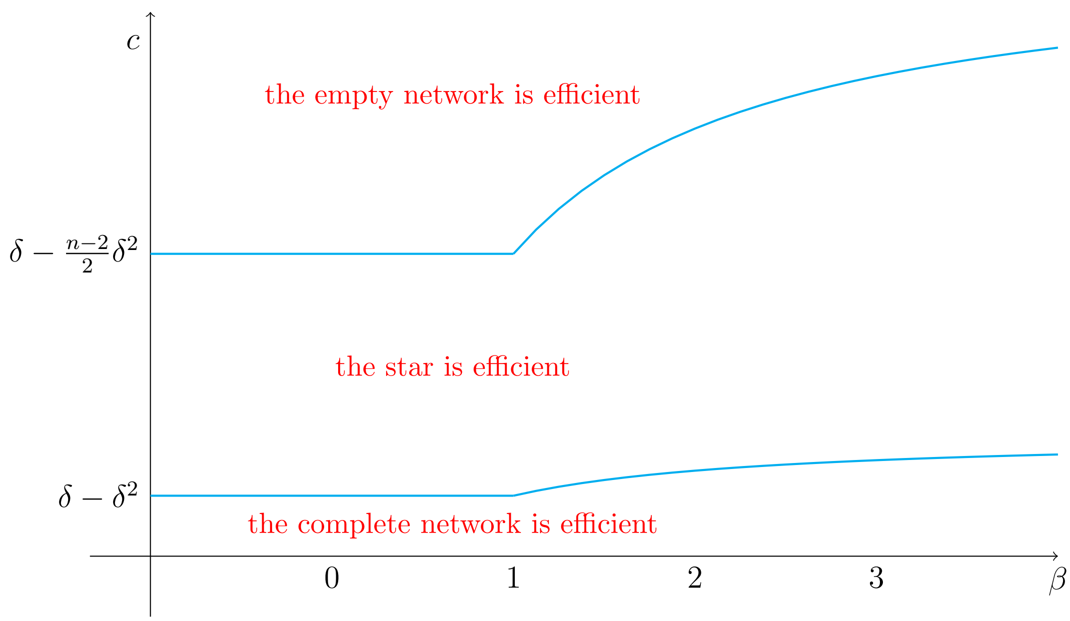


Figure 2: Range of c that makes the empty/star/complete network efficient

[8] The intuition for why the efficient network is the complete network when c is small, the star network when c is in an intermediate range, and the empty network when c is large is as explained in Jackson and Wolinsky (1996): The complete network and the empty network are naturally efficient for small and large c respectively. The star network is efficient for intermediate c because it connects and brings players as close as possible with the least number of links.

Notice that as β approaches ∞, $g_{ij}(a) = \mathbb{1}\{\max\{a_{ij}, a_{ji}\} \geq 1\}$, hence links will always be unilaterally sponsored by players. This is the assumption adopted by Bala and Goyal (2000) where the connections model is also investigated. Our efficiency characterization when $\beta \to \infty$ is unsurprising identical to theirs. Proposition 1 therefore informs us how we transit from the efficiency characterization under the bilateral link formation assumption in Jackson and Wolinsky (1996) to that under the unilateral link formation assumption in Bala and Goyal (2000) along the dimension of link investment substtutability: While the characterization in Jackson and Wolinsky (1996) is robust for all situations when link investments are strategic complements, the efficient network tends to be denser as β rises until it reaches the specification in Bala and Goyal (2000).

We now move to characterize the weighted stable networks.

To start with, we analyze the incentive of a player to build, maintain, or sever a link. To do so, we define and derive the *marginal connection benefit* and the *marginal investment cost* of a link for a player.

The *marginal connection benefit* player i obtains by linking with player j in network g is the difference in the amount of intrinsic value i obtains through the network with and without a link to j:

$$MB(i \leftarrow j, g) = \begin{cases} \sum_{k \in N} \delta^{d_{ik}(g+ij)} - \sum_{k \in N} \delta^{d_{ik}(g)} & \text{if } g_{ij} = 0 \\ \sum_{k \in N} \delta^{d_{ik}(g)} - \sum_{k \in N} \delta^{d_{ik}(g-ij)} & \text{if } g_{ij} = 1 \end{cases}$$

The *marginal investment cost* player i pays for a link with player j under an investment profile a is simply the cost of investment a_{ij}:

$$MC(i \to j, a) = ca_{ij}$$

A player i decides whether to have a link with j by trading off the marginal connection benefit with the marginal investment cost. We can see that while the representation of $MC(i \to j, a)$ is straight-forward, that of $MB(i \leftarrow j, g)$ is less so. In what follows, we explore how $MB(i \leftarrow j, g)$ is determined by network structure g.

To assess $MB(i \leftarrow j, g)$, we need to know how i's distances to others change with an addition or deletion of link with j. When a link with j is added (deleted), i not only reduces (increases) his distance to j to 1 (at least 2), but also reduces (increases) distances to other players whom j is relatively closer to. This effect can be captured with information on i's, j's and i's neighbours' distances to other players in network g. Let $d_i(g) = \{d_{i1}(g), ..., d_{in}(g)\}$ be the *distance vector* of player i in network g, as follows.

Lemma 1. *For players $i \in N$, $j \neq i$ and $k \neq i$:*

(i)

$$d_{ik}(g + ij) = \begin{cases} d_{jk}(g) + 1 & \textit{when } d_{ik}(g) > d_{jk}(g) \\ d_{ik}(g) & \textit{otherwise} \end{cases}$$

(ii)

$$d_{ik}(g-ij)=\begin{cases}\min\limits_{l:l\neq j,g_{il}=1} d_{lk}(g)+1 & \textit{when } d_{ik}(g)>d_{jk}(g)\\ d_{ik}(g) & \textit{otherwise}\end{cases}$$

The intuition behind Lemma 1 is simple. If i gets a new link with j, i's distance to other players may shorten as there is now a new path from i to them via j. Whether the new path is at least as short as all the original ones depends on if $d_{jk}(g) < d_{ik}(g)$. If it is, the new distance will be $d_{jk}(g)+1$; if not, the distance remains $d_{ik}(g)$. Similarly, if i cuts his link with j, i's distance to other players may lengthen as he loses a path to them via j. Whether this is the case depends on if the original shortest path between i and another player passes through j, again indicated by whether $d_{jk}(g) < d_{ik}(g)$. If it is, the new distance will be the shortest path from i to the other node without passing j, which is $\min\limits_{l:l\neq j,g_{il}=1} d_{lk}(g)+1$; if not, the distance remains $d_{ik}(g)$.

With Lemma 1, we represent the marginal connection benefit as:

$$MB(i\leftarrow j,g)=\begin{cases}\sum\limits_{k\in N:d_{ik}(g)>d_{jk}(g)}\left(\delta^{d_{jk}(g)+1}-\delta^{d_{ik}(g)}\right) & \text{if } g_{ij}=0\\ \sum\limits_{k\in N:d_{ik}(g)>d_{jk}(g)}\left(\delta^{d_{ik}(g)}-\delta^{\min\limits_{l\in N:l\neq j,g_{il}(g)=1} d_{lk}(g)+1}\right) & \text{if } g_{ij}=1\end{cases}$$

We can see that $MB(i \leftarrow j, g)$ is determined by how many players j has relatively closer access to than i does, and how much closer j is compared to i.

Going back to the question of whether i wants to build or sever a link with j. Since $MC(i \to j, a) = ca_{ij}$, we compare $MB(i \leftarrow j, g)/c$ and a_{ij}.

We know that a link between i and j is created and sustained if $h(a_{ij}, a_{ji}) = (\frac{1}{2}a_{ij}^{\beta} + \frac{1}{2}a_{ji}^{\beta})^{\frac{1}{\beta}} \geq 1$. So, there is a way of link sponsorship where both i and j are willing to maintain the link if we can find a pair of a_{ij} and a_{ji} such that $\frac{MB(i\leftarrow j,g)}{c} \geq a_{ij}$, $\frac{MB(j\leftarrow i,g)}{c} \geq a_{ji}$, and $h(a_{ij}, a_{ji}) \geq 1$, which means

$$h(\frac{MB(i\leftarrow j,g)}{c},\frac{MB(j\leftarrow i,g)}{c})\geq 1$$

and there is no way of link sponsorship where both i and j find creating a link beneficial (one finds if strictly beneficial) if

$$h(\frac{MB(i\leftarrow j,g)}{c},\frac{MB(j\leftarrow i,g)}{c})\leq 1$$

Summarizing our observations, we arrive at Proposition 2.

Proposition 2. *Consider the weighted connections model. A network $g \in G$ is weighted pairwise stable if and only if for all pairs of players $(i, j) \in N^2$:*

(i) if $g_{ij} = 1$*, then*

$$h(\frac{MB(i \leftarrow j, g)}{c}, \frac{MB(j \leftarrow i, g)}{c}) \geq 1$$

(ii) if $g_{ij} = 0$*, then*

$$h(\frac{MB(i \leftarrow j, g)}{c}, \frac{MB(j \leftarrow i, g)}{c}) \leq 1$$

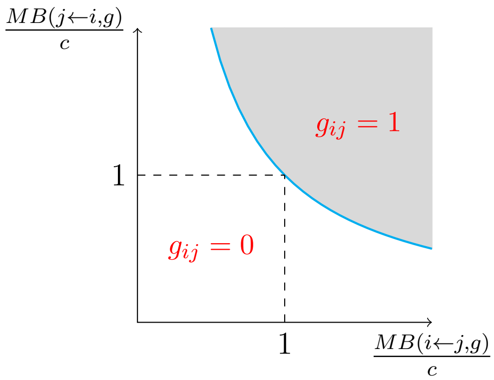
where

$$MB(i \leftarrow j, g) = \begin{cases} \sum_{k \in N: d_{ik}(g) > d_{jk}(g)} \left(\delta^{d_{jk}(g)+1} - \delta^{d_{ik}(g)}\right) & \text{if } g_{ij} = 0 \\ \sum_{k \in N: d_{ik}(g) > d_{jk}(g)} \left(\delta^{d_{ik}(g)} - \delta^{\min\limits_{l \in N: l \neq j, g_{il}(g)=1} d_{lk}(g)+1}\right) & \text{if } g_{ij} = 1 \end{cases}$$

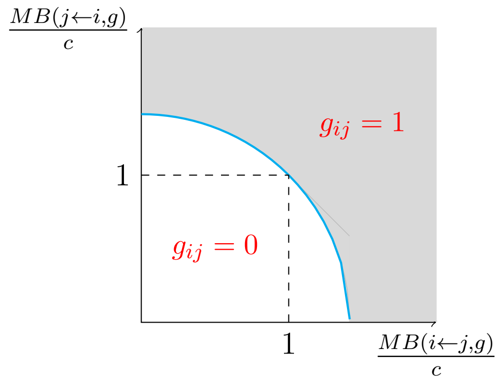
Note that with different levels of β, the shape of the h function changes. Therefore, the level of link investment substitutability influences players' decision on when to keep and form links and hence affect the stability of networks. It can be shown the for the CES function h, when $\beta_1 \geq \beta_2$, it is always the case that

$$h(\frac{MB(i \leftarrow j, g)}{c}, \frac{MB(j \leftarrow i, g)}{c})|_{\beta=\beta_1} \geq h(\frac{MB(i \leftarrow j, g)}{c}, \frac{MB(j \leftarrow i, g)}{c})|_{\beta=\beta_2}$$

So in a network g, if player i and player j want to maintain or form a link under $\beta = \beta_2$, they would also want to do so under $\beta = \beta_1$. A greater link investment substitutability increases the chances for two players to have a link between them. Figure 3 provides a simple graphic illustration for this effect.



(a) when $\beta = 0$



(b) when $\beta = 2$

Figure 3: Combinations of marginal connection benefits that supports a link

The blue curves in Figures 3(a) and 3(b) plot the combinations of marginal connection benefits such that $h(\frac{MB(i \leftarrow j,g)}{c}, \frac{MB(j \leftarrow i,g)}{c}) = 1$ when $\beta = 0$ and $\beta = 2$ respectively. When the marginal connection benefits i and j provide to each other lie in the grey area, i and j would maintain their link if $g_{ij} = 1$ and create a link if $g_{ij} = 0$. By comparing Figure 3(a) and Figure 3(b), we can see a greater β leads to a larger range of marginal connection benefit pairs that facilitate a link. In particular, if one of two players enjoys a large marginal connection benefit from a link while the other gets a small marginal connection benefit from the link, they can only be linked when β is large.

Recall that in the connections model, the marginal connection benefit of a link is determined by the two players' relative distance to other players, different link investment substitutability thus imply different network positions of a pair of linked/unlinked players in a stable network. When β is small, for two players i and j to have a link, they both need to provide certain amount of marginal connection benefit to the other: i needs to have shorter distances to some players than j does and j needs to have shorter distances to some other players than i does. It is not likely that one of two linked players is isolated from others. When β is large, two players can have a link as long as one enjoys a large marginal connection benefit from the link. A link a poorly-connected player is now common provided that a well-connected player is on the other side of the link.

We illustrate the above implication of β on stable networks with an examination of the stability of the star network. Since links in the star network are all between a very well-connected centre player and a relatively poorly-connected periphery player, according to our analysis, we would expect the star network to be weighted pairwise stable when β large but not so when β is small. We verify this conjecture below.

Example 1 (Weighted Pairwise Stability of the Star Network under Different Link Investment Substitutability)**.** For the star network to be weighted pairwise stable, we require that any pair of centre player m and periphery player p do not want to sever their link, and any pair of periphery players p and p' do want want to form a link:

$$\begin{cases} h(\frac{MB(m \leftarrow p,star)}{c}, \frac{MB(p \leftarrow m,star)}{c}) \geq 1 \\ h(\frac{MB(p \leftarrow p',star)}{c}, \frac{MB(p' \leftarrow p,star)}{c}) \leq 1 \end{cases}$$

For a star network with n players, if the centre player cuts his link with a periphery player, his distances to other players are not changed and he only forgoes the exchange of intrinsic value with the periphery player that worth δ, so

$$MB(m \leftarrow p, star) = \delta$$

If a periphery player loses his link with the centre player, he loses connections to all other players. Since his distance to the centre player was 1 and his distances to the other $(n-2)$ periphery players were 2,

$$MB(p \leftarrow m, star) = \delta + (n-2)\delta^2$$

Finally, if two periphery players form a link, they shorten the distance to each other. But their distances to other players are not changed, hence

$$MB(p \leftarrow p', star) = MB(p' \leftarrow p, star) = \delta - \delta^2$$

Plug the expressions of marginal connection benefits to the conditions for the star to be weighted pairwise stable, we can see that the star is pairwise stable if and only if

$$\begin{cases} \left(\frac{1}{2}(\frac{\delta}{c})^\beta + \frac{1}{2}(\frac{\delta+(n-2)\delta^2}{c})^\beta\right)^{\frac{1}{\beta}} \geq 1 \\ \left(\frac{1}{2}(\frac{\delta-\delta^2}{c})^\beta + \frac{1}{2}(\frac{\delta-\delta^2}{c})^\beta\right)^{\frac{1}{\beta}} \leq 1 \end{cases}$$

which can be simplified to

$$\delta - \delta^2 \leq c \leq \left(\frac{1}{2}\delta^\beta + \frac{1}{2}(\delta + (n-2)\delta^2)^\beta\right)^{\frac{1}{\beta}}$$

As we expect, the range of link investment cost c that makes the star weighted pairwise stable changes with the value of β. Figure 4 depicts this change for a scenario with 12 players and a discount factor of 0.8. We can see that the range of c that supports the star significantly broadens as β gets larger. Greater link investment substitutability supports the star structure.

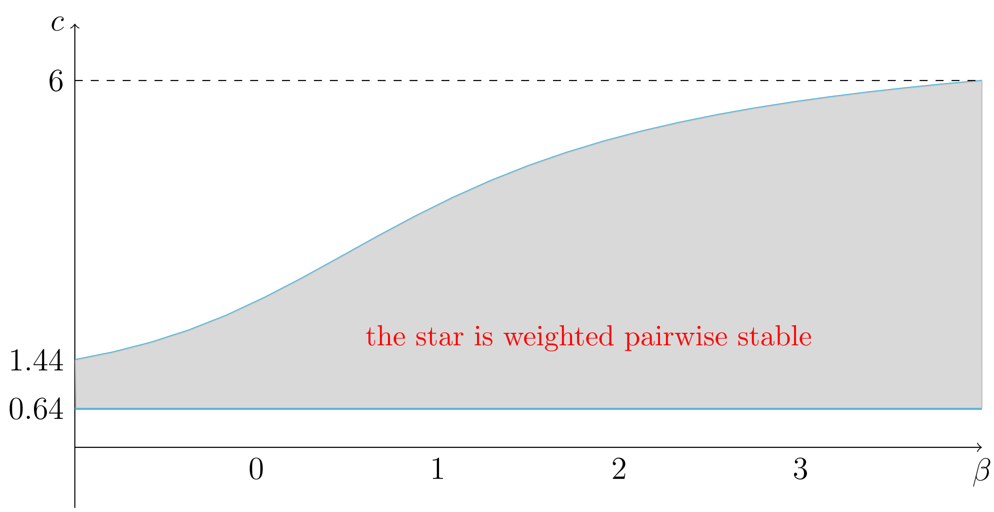


Figure 4: Range of c that makes the star weighted pairwise stable ($\delta = 0.8, n = 12$)

We finish our analysis for the weighted connections model with an investigation on the tension between efficiency and stability. We measure the tension in the following way. For a network g that can be efficient, we compare the range of situations in which it is efficient but not stable with the range of situations in which it is efficient and obtain a ratio between 0 and 1 that captures the likelihood for an efficient network not to be stable. The larger the ratio, the stronger the tension.

Formally, for the measure to be well-defined, we restrict the cost of link investment to be between 0 and n: $c \in [0, n]$. Note that if $c > n$, the empty network is efficient and is the unique weighted pairwise stable network, so analysis for situations where c is beyond n does not shed much light on our understanding of the connections model. Let $C^e(g; \beta, \delta) \subset [0, n]$ be the set of link investment costs such that network g is efficient under β and δ, and $C^{wps}(g; \beta, \delta) \subset [0, n]$ be the set of link investment costs such that network g is weighed pairwise stable under β and δ. We define, for $g \in G$ that can be efficient for some $c \in [0, n]$,

$$T(g; \beta, \delta) = 1 - \frac{| C^e(g; \beta, \delta) \cap C^{wps}(g; \beta, \delta) |}{| C^{wps}(g; \beta, \delta) |}$$

We show that

Proposition 3. *Consider the weighted connections model. For any distant factor δ,*

(i) T(complete; β, δ) equals to 0 when $\beta \leq 1$ and increases with β when $\beta > 1$,

(ii) T(star; β, δ) decreases with β when $\beta \leq 1$, reaches 0 when $\beta = 1$, and increases with β when $\beta > 1$,

(iii) T(empty; β, δ) equals to 0 for all $\beta \in \mathbb{R}$

First, note that T(*empty*; $\beta, \delta) = 0$ for all β and δ. This is because a link between two players always creates positive externality to others by facilitating indirect connections. So, if the empty network is efficient, meaning that it is optimal for players not to form links taken the positive extnerlity into account, it must be that players would not want to form links for their own utility maximization, making the emoty network also stable.

Turning out attension to situations where the complete network or the star network is efficient, Proposition 3 shows when $\beta = 1$, T$(g; \beta, \delta) = 0$ for both networks and all discount factors, indicating that there is no tension between efficiency and stability when link investments are perfect substitutes. The intuition behind the result is the following. When an efficient network is not stable, there is a pair of players who do not find it mutually profitable to form a link that enhances aggregate welfare. A possible cause for the failure of link formation is the inflexibility of the link formation protocal. For example, under the bilateral link formation specification, if i benefits significantly from a link with j but j only benefits trivially from the link, the link cannot be formed since i and j need to make the same level of link investments. When link investments become more substitutable so that links can be sponsored in more flexible manners, the above restriction is relaxed: i and j can coordinate an unequal sponsorship of the link where i invests more and j invests less. When link investments are perfect substitutes, players can essentially make transfers to each other and the tension between efficiency and stability can be completely eliminated.

Like Proposition 1, Proposition 3 also serves to bridge results from Jackson and Wolinsky (1996) and Bala and Goyal (2000). We already know that Jackson and Wolinsky (1996) shows a great tension between efficiency and stability. In Bala and Goyal (2000) where they adopt the unilateral link formation assumption, the inconsistency is much reduced. From Proposition 3, we learn that the tension starts strong when $\beta \to -\infty$, decreases to 0 when β reaches 1, and the rises slightly as β grows and approaches $+\infty$. The role of β on welfare properties of network formation is not monotonic.

4 Application II: Law of the Few

The second network formation scenario we examine is one where players produce goods themselves and simultaneously form links to enjoy the externalities from the production of others. Galeotti and Goyal (2010) model such interactions with the assumption that link investments are binary and links can be unilaterally formed. They solve for the Nash equilibrium and strict Nash equilibrium of the game and show that in equilibrium, only a small fraction of players engage in production activities while the majority of players focus on link formation. They use the results to explain the empirical finding that in many online information sharing networks, the number of users writing comments and reviews is small and most users only follow the content providers. Based on this finding, they title their paper *Law of the Few*.

We extend the model in Galeotti and Goyal (2010) by allowing weighted link investments and assume that links can be formed with the CES link formation technology. We carry out this exercise because we believe in certain contexts, our assumption better describes the situations players face. For example, when information sharing is achieved with email exchanges, players can choose the frequency of correspondence and they enjoy better communication efficiency when the two sides put in a similar amount of effort. When information sharing is achieved with face to face contacts, players can again choose how enthusiastic to be and the complementarity between link investments can be even greater than in the case of email exchanges. Allowing weighted link investments is a natural modelling choice and we use the CES specification to study how different levels of link investment substitutability affects the communication structure.

We show that the law of the few prediction in Galeotti and Goyal (2010) is robust when link investments are strategic substitutes (when $\beta > 1$), but the equilibrium characterization becomes quite different when link investments get more complementary. In that case, all players make some production effort on their own, links are sponsored in relatively reciprocal ways, and there is less disparity in the network positions of players. We also study the welfare of players under different levels of link investment substitutability.

In the following parts of the analysis, we describe the model and results with an information searching and sharing story. However, the understanding obtained cab ve applied to all contexts where the access to a local public good is determined by a network that is endogeneously formed.

4.1 The Model

Consider the following game played by a set of players $N = \{1, ..., n\}$. Each player $i \in N$ chooses $s_i = \{x_i, a_i\}$, where $x_i \in X$ represents player i's effort in searching and $a_i = \{a_{i1}, ..., a_{ii-1}, a_{ii+1}, ...a_{in}\} \in A_i$ represents i's investments in links with other players. We assume the space of strategies player i can choose from to be $S_i = X \times A_i = \mathbb{R}^n$. A strategy profile is $s = \{s_1, ..., s_n\}$ and the space of strategy profiles is $S = S_1 \times S_2 \times ... \times S_n$.

The link investments of all players $a = \{a_1, ..., a_n\}$ pin down a network structure (N, g) where $g = \{g_{ij}\}_{i,j\in N}$ is an $n \times n$ matrix and g_{ij} measures the strength of link between i and j. We assume that g_{ij} is determined by a_{ij} and a_{ji} according to the CES link formation specification:

$$g_{ij}(a) = h(a_{ij}, a_{ji}) = (\frac{1}{2}a_{ij}^{\beta} + \frac{1}{2}a_{ji}^{\beta})^{\frac{1}{\beta}}$$

where $\beta \in \mathbb{R}$ captures the degree of substitutability between link investments a_{ij} and a_{ji}.

We define the utility of player i under strategy profile $s \in S$ as:

$$u_i(s) = f(x_i + \sum_{j\neq i} g_{ij}(a)x_j) - cx_i - k\sum_{j\neq i} a_{ij}$$

where f is twice continuously differentiable, increasing, and strictly concave, $c > 0$ and $k > 0$. The utility specification is based on the following story. Players get utility from information that comes from searching or communication with other players they have links with. The proportion of information one can get from another player is proportional to the strength of link they have. There is a cost $c > 0$ for effort and a cost $k > 0$ for link investment. The f function captures how players value information: the value is increasing with information at a decreasing rate. Assume that $f'(0) > c$ and $\lim_{y\to+\infty} f'(y) < c$ so there exists a $\hat{y} > 0$ such that $f'(\hat{y}) = c$. Consider a player who does not have links, $\hat{y}$ would be his optimal effort level and the resulting amount of information he obtains.

We solve for the Nash equilibrium of the game. A strategy profile s^* is a Nash equilibrium if for all $i \in N$ and all $s_i \in S_i$:

$$u_i(s^*) \geq u_i(s_i, s_{-i}^*).$$

4.2 Equilibrium Characterization

We first characterize the Nash equilibrium of the game when link investments are strategic complements (when $\beta < 1$). In this case, the equilibrium network can consist of several components where a pair of players are connected if and only if they belong to the same component.

We now define the concept of a component. For the purpose of analysing our model, we also include information on efforts and link investments into our definition.

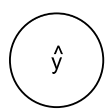
Definition 4. Two players i and j are connected in a network g if there exists a sequence of nodes $i = v_0, v_1, v_2, ..., v_m = j$ such that $g_{v_{k-1}v_k} > 0$ for $k = 1, ..., m$.

A component $C = (N^C, s^C)$ of a strategy profile $s = \{x, a\}$ is a subset of players and their strategies such that $\emptyset \neq N^C \subset N$, $s^C = \{s_i\}_{i \in N^C}$, and

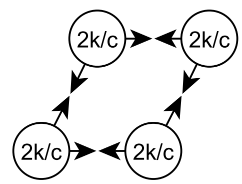
(i) $\forall i, j \in N^C$: i and j are connected in $g(a)$,

(ii) $\forall i \in N^C, j \notin N^C$: i and j are not connected in $g(a)$.

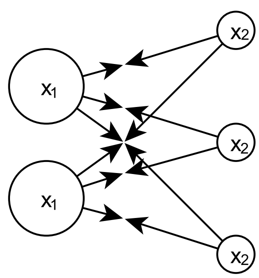
Our equilibrium characterization involves four kinds of components: the *isolated component*, the *regular component*, the *bipartite component* and the *multi-centre star component*.



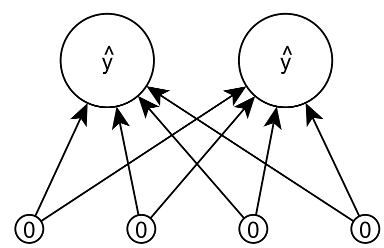
(a) the isolated component I



(b) the regular component R



(c) the bipartite component B



(d) the multi-centre star component S

Figure 5: Four kinds of equilibrium components

To be specific, the isolated component $I = (N^I, s^I)$ has a single player who makes $\hat{y}$ amount of effort and does not invest in relationships with others:

$$\begin{cases} | N^I |= 1 \\ \forall i \in N^I : x_i = \hat{y} \text{ and } a_{ij} = 0 \text{ for all} j \neq i \end{cases}$$

The regular component $R = (N^R, s^R)$ has a set of players who all exert $2k/c$ amount of effort, make $c\hat{y}/2k - 1$ amount of link investments to other players in the component, and sponsor links to each other in a completely reciprocal manner:

$$\begin{cases} \mid N^R \mid > 1 \\ \forall i \in N^R : x_i = \frac{2k}{c} \text{ and } \sum a_{ij} = \frac{c\hat{y}}{2k} - 1 \text{ where } a_{ij} > 0 \text{ only if } j \in N^R \\ \forall i, j \in N^R : a_{ij} = a_{ji} \end{cases}$$

The bipartite component $B = (N^B, s^B)$ has two groups of players exerting different amounts (x_1 and x_2) of effort and investing in intergroup links. The way a link between two players is sponsored depends on the relative amount of efforts they make. Let x_1 and x_2 be two scalers where $x_1, x_2 \in (0, \hat{y})$, $x_1 \neq x_2$ and

$$x_1^{\frac{\beta}{1-\beta}} + x_2^{\frac{\beta}{1-\beta}} = 2^{\frac{1}{1-\beta}} (\frac{k}{c})^{\frac{\beta}{1-\beta}}$$

The bipartite component features:

$$\begin{cases} N^B = U \cup V \text{ where } U \neq \emptyset, V \neq \emptyset, \text{ and } U \cap V = \emptyset \\ \forall i \in U : x_i = x_1 \text{ and } \sum a_{ij} = (\frac{cx_2^\beta}{2k})^{\frac{1}{1-\beta}} (\hat{y} - x_1) \text{ where } a_{ij} > 0 \text{ only if } j \in V \\ \forall j \in V : x_j = x_2 \text{ and } \sum a_{ji} = (\frac{cx_1^\beta}{2k})^{\frac{1}{1-\beta}} (\hat{y} - x_2) \text{ where } a_{ij} > 0 \text{ only if } j \in U \\ \forall i \in U, j \in V : \frac{a_{ij}}{a_{ji}} = (\frac{x_2}{x_1})^{\frac{1}{1-\beta}} \text{ or } a_{ij} = a_{ji} = 0 \end{cases}$$

The multi-centre star component $S = (N^S, s^S)$ has two groups of players where players in one group make $\hat{y}$ amount of effort while players in the other group do not make effort and unilaterally sponsor links to effort makers:

$$\begin{cases} N^S = U' \cup V' \text{ where } U' \neq \emptyset, V' \neq \emptyset, \text{ and } U' \cap V' = \emptyset \\ \forall i \in U' : x_i = \hat{y} \text{ and } a_{ij} = 0 \text{ for all} j \neq i \\ \forall j \in V' : x_j = 0 \text{ and } \sum a_{ji} = 2^{\frac{1}{\beta}} \frac{\hat{z}}{\hat{y}} \text{ where } \hat{z} \text{ satisfies } f'(\hat{z}) = 2^{\frac{1}{\beta}} \frac{k}{\hat{y}} \text{ and } a_{ij} > 0 \text{ only if } j \in U' \end{cases}$$

We have the following Nash equilibrium characterization of the game when link investments are strategic complements.

Proposition 4. *Consider the game with $\beta < 1$. If $\hat{y} \leq 2k/c$, the game has a unique Nash equilibrium where each player forms an I component. If $\hat{y} > 2k/c$, then s^* is an equilibrium if and only if the components players form are*

(i) a combination of I, R and B components when $\beta < \frac{ln2}{ln(c\hat{y}/k)}$,

(ii) a combination of I, R, B and S components when $\beta = \frac{ln2}{ln(c\hat{y}/k)}$,

(iii) a combination of R and B components, or a single S component with $\mid V' \mid = 1$ when $\frac{ln2}{ln(c\hat{y}/k)} < \beta < 1$.

It is easy to see that when $\hat{y} \leq 2k/c$, there will not be any linking activity in equilibrium because the amount information one can get from a neighbour is too small compared to the cost of a link. We are more interested in understanding the equilibrium when $\hat{y} > 2k/c$ such that there can be some linking activity. We make two general observations for this case. First, there can always be a very symmetric equilibrium structure that consists only of regular components. Second, an asymmetric equilibrium structure featuring the multi-centre star can emerge as β grows over a certain threshold (link investments are sufficiently substitutable).

Another thing to note is that link investment substitutability influences the structure of bipartite components that can be formed in equilibrium. From our previous definition of the bipartite component, we can see that β affects the relationship between the levels of efforts made by players from the two groups. It also determines the relative link investments from two linked players given their effort levels.

We illustrate the influence of β on efforts with Figure 6 that plots possible values of x_1 and x_2 under two different βs.

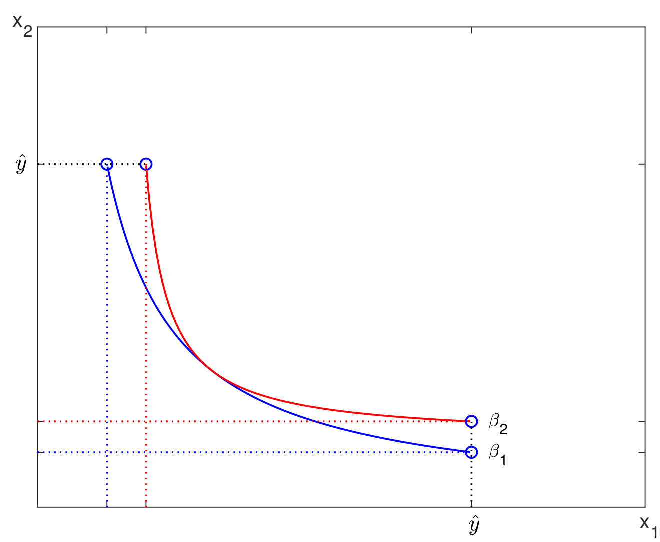


Figure 6: Possible effort levels of two linked players when $\beta = \beta_1$ and when $\beta = \beta_2$ ($\beta_1 > \beta_2$)

First note that with both βs, when a player from one group of the bipartite component makes more efforts, his linked player from the other group makes less. This is naturally so since information searched on one's own and information obtained from neighbours are strategic substitutes. Link investment substitutability plays a role in affecting how much the effort a player in one group makes drops as the player he is linked to in the other group searches more. We can see that a greater β (β_1) leads to a greater change and hence a larger possible effort level disparity between two groups.

Regarding the effect of β on link investments, we know that

$$\frac{a_{ij}}{a_{ji}} = (\frac{x_2}{x_1})^{\frac{1}{1-\beta}} = (\frac{x_j}{x_i})^{\frac{1}{1-\beta}}$$

and $\frac{1}{1-\beta} > 0$ when $\beta < 1$. So, a player i invests relatively more than j does to their link if j searches relatively more. The level of link investment substitutability determines the scale of the influence relative efforts have in relative link investments. Since $\frac{1}{1-\beta}$ rises with β, a greater link investment substitutability leads to a more elastic response in link investments to differences in efforts.

In Galeotti and Goyal (2010), they discuss how requiring mutual consents for link formation would affect the stable communication sturcture. It turns out that then all players must acquire some information personally and they show that the regular network is stable for a wide range of parameters. Our complete equilibrium characterization for all $\beta < 1$ shows that their first description remains true as long as the link investment substitutability is not too large ($\beta < \frac{ln2}{ln(c\hat{y}/k)}$) an their second prediction holds for all cases when link investments are strategic complements. We also demonstrate that other than the regular components, bipartite components can often emerge when link investments are strategic complements.

We now move to investigate the Nash equilibrium of the game when link investments are strategic substitutes (when $\beta > 1$).[9] We show that a Nash equilibrium must have the following features.

Proposition 5. *Consider the game with $\beta > 1$. If $\hat{y} < 2^{\frac{1}{\beta}}k/c$, the game has a unique Nash equilibrium where each player forms an I component. If $\hat{y} \geq 2^{\frac{1}{\beta}}k/c$, then a single S component with $\mid V' \mid = 1$ is always a Nash equilibrium.*

A Nash equilibrium s^* when $\hat{y} \geq 2^{\frac{1}{\beta}}k/c$ must satisfy (i) the effort level of all players lies between $[0, \hat{y}]$ and there can be at most one player who makes for more than $2^{\frac{1}{\beta}}k/c$ amount of effort, and (ii) all links are sponsored unilaterally and only players who search for at least $2^{\frac{1}{\beta}}k/c$ receive link investments.

Again, when $\hat{y}$ is small, players do not invest in links because the amount of information they can obtain via a link is too small to incentivize investments. When $\hat{y}$ exceeds the $2^{\frac{1}{\beta}}k/c$ threshold, players start to make link investments. Note that equilibrium structures here are very different from those that can emerge when link investments are strategic complements. The most obvious distinction is that links must be sponsored unilaterally when link investments are strategic substitutes, which contradicts the reciprocal link investments in the regular components and the proportional link investments in the bipartite components that can always emerge when link investments are strategic complements.

[9] Equilibrium properties when $\beta = 1$ is similar to when $\beta > 1$. The only difference lies in condition (ii) of Proposition 5: Links are not necessarily unilaterally sponsored when $\beta = 1$. The proof for this statement can be found in the Appendix.

Also, note that now the equilibrium network has a core-periphery structure where players who search for at least $2^{\frac{1}{\beta}}k/c$ are in the core and can have links to each other while players who search for less than $2^{\frac{1}{\beta}}k/c$ are in the periphery and issue links to core players. We also show that there can only be one player who searches for a large amount of (more than $2^{\frac{1}{\beta}}k/c$) information.

Our Nash equilibrium characterization with $\beta > 1$ resonates with the equilibrium characterization in Galeotti and Goyal (2010). They show that under the unilateral link investment assumption ($A_{ij} \in \{0, 1\}$ and $g_{ij} = \max\{a_{ij}, a_{ji}\}$ for all $i \in N, j \neq i$), the Nash equilibrium network has a core-periphery structure where core players search for more information than periphery players.

With equilibrium analysis for all $\beta \in \mathbb{R}$, we fill the gap in Galeotti and Goyal (2010) where they provide characterizations for when $\beta \to +\infty$ and when $\beta \to -\infty$ respectively. In Galeotti and Goyal (2010), characterizations under the unilateral and the bilateral link formation assumptions are very different. We show that strategic substitutability/complementarity of link investments is key to understand the different in the characterization. On top of this, we demonstrate that when link investments are strategic complements, the level of link investment substitutability affects specific features of equilibrium structures. More detailed analysis on the influence of link investment substitutability is given in Section 4.3.

4.3 Comparative Statics

We provide comparative statics analysis on β in this section. We ask ourselves two questions. First, as analysed in the previous section, how does β influence the distribution of efforts and link investments? Second, in an asymmetric equilibrium where players choose different strategies, their utilities might be different and depends on how heavily they rely on information from others. How does β affect this relationship?

For the first question, we derive the supremum and infimum disparity in equilibrium efforts and mutual link investments. These measures inform us of how specialized the society is. Formally, let $S^*(\beta, f, k, c)$ be the set of equilibrium for the game with parameters β, k, c and funtion f, we derive

$$SD_x(\beta, f, k, c) = \sup_{s^* \in S^*(\beta,f,k,c), i,j \in N, j \neq i} \left|\frac{x_i^* - x_j^*}{x_i^* + x_j^*}\right|$$

$$ID_x(\beta, f, k, c) = \inf_{s^* \in S^*(\beta,f,k,c), i,j \in N, j \neq i} \left|\frac{x_i^* - x_j^*}{x_i^* + x_j^*}\right|$$

$$SD_a(\beta, f, k, c) = \sup_{s^* \in S^*(\beta,f,k,c), i,j \in N, g_{ij}* > 0} \left|\frac{a_{ij}^* - a_{ji}^*}{a_{ij}^* + a_{ji}^*}\right|$$

$$ID_a(\beta, f, k, c) = \inf_{s^* \in S^*(\beta,f,k,c), i,j \in N, g_{ij}* > 0} \left|\frac{a_{ij}^* - a_{ji}^*}{a_{ij}^* + a_{ji}^*}\right|$$

where SD and ID are short for supremum difference and infimum difference respectively and the subscriptes x and a indicate whether the measure is for effort or link investment. All measures are normalized so that they lie between 0 and 1.

We show that all these measures of disparity are non-decreasing in β.

Proposition 6. *Consider the game with $\hat{y} > 2k/c$ so that there always exists an equilibrium with positive link investment. For any f, k and c,*

(i) $SD_x(\beta, f, k, c)$ rises with β when $\beta < \frac{ln(2)}{ln(c\hat{y}/k)}$ and equals to 1 when $\beta \geq \frac{ln(2)}{ln(c\hat{y}/k)}$,

(ii) $ID_x(\beta, f, k, c)$ equals to 0 for all $\beta \in \mathbb{R}$,

(iii) $SD_a(\beta, f, k, c)$ rises with β when $\beta < \frac{ln(2)}{ln(c\hat{y}/k)}$ and equals to 1 when $\beta \geq \frac{ln(2)}{ln(c\hat{y}/k)}$,

(iv) $ID_a(\beta, f, k, c)$ equals to 0 for when $\beta \leq 1$ and equals to 1 when $\beta > 1$.

From Proposition 6, we can see that the supremum difference in equilibrium efforts and in equilibrium link investments are affected by β in a similar pattern. They both rise with β and reach 1 when β arrives at $\frac{ln(2)}{ln(c\hat{y}/k)}$, indicating that it is possible to observe greater inequality in effort levels and link investments as link investment substitutability rises and the inequality can take its greatest form when β gets larger than $\frac{ln(2)}{ln(c\hat{y}/k)}$.

For the infimum difference measures, while $ID_x(\beta, f, k, c)$ remains 0 for all values of β, $ID_a(\beta, f, k, c)$ takes a huge jump from 0 to 1 when link investments change from strategic complements ($\beta < 1$) to strategic substitutes ($\beta > 1$). This is because when link investments are strategic complements, links are always reciprocally sponsored in a regular component that can emerge in equilibrium. But when link investments are strategic substitutes, links are always unilaterally sponsored.

We have learnt from Proposition 6 how different characteristics of equilibrium structure change smoothly or discretely with β. We now move to the second question on how β affects the utility distribution of players.

In an asymmetric equilibrium, players adopt different strategies to obtain information. Some can specialize in searching while others devote more to linking. The resulting utility of players are different given different equilibrium strategies. We analyse the relationship between players' behaviour (their level of effort) and their utility and how β affects this relationship.

Proposition 7. *Let s^* be a Nash equilibrium of the game. When $\beta < 1$, $0 < x_i^* < x_j^* < \beta\hat{y}$ or $\hat{y} > x_i^* > x_j^* > \beta\hat{y}$ implies $u_i(s^*) < u_j(s^*)$. When $\beta > 1$, $x_i^*, x_j^* \in [0, 2^{\frac{1}{\beta}}k/c)$ implies $u_i(s^*) = u_j(s^*)$; $x_i^* \in [0, 2^{\frac{1}{\beta}}k/c)$ and $x_j^* = 2^{\frac{1}{\beta}}k/c$ imply $u_i(s^*) \leq u_j(s^*)$; and $x_i^* \in [0, 2^{\frac{1}{\beta}}k/c]$ and $x_j^* > 2^{\frac{1}{\beta}}k/c$ imply $u_i(s^*) > u_j(s^*)$.*

Proposition 7 tells us that when $\beta < 1$ so that link investments are strategic complements, there is an optimal effort level of $\max\{0, \beta\hat{y}\}$ that leads to the highest equilibrium utility: When two players both search for less than $\beta\hat{y}$ or more than $\beta\hat{y}$, the one whose effort is closer to $\beta\hat{y}$ obtains a larger utility. Since this optimal effort level rises with β, greater link investment substitutability favours those who put in more searching effort. The reason for this favour is that with greater link investment substitutability, searchers can attract more link investments from other players and then access their information at a lower cost.

When link investments are strategic substitutes, Proposition 7 shows an advantage for a player who searches for $2^{\frac{1}{\beta}}k/c$ but a disadvantage for a player who searches for more than $2^{\frac{1}{\beta}}k/c$. The advantage for those who make $2^{\frac{1}{\beta}}k/c$ amount of effort can be explained with the same reasoning above on why greater β favours searchers. When a player searches for $2^{\frac{1}{\beta}}k/c$, he will receive unilateral link investments from other players who might be searching. He hence accesses some information for free. However, if a player searches for more than $2^{\frac{1}{\beta}}k/c$, he will also receive unilateral link investments but those who invest will not search at all since acquiring information from the player is cheaper than searching. He hence obtains no information from his links and only get free-rode by his neighbours.

5 Discussion

This paper suggests a research agenda on link formation technology and network formation. We have proposed an approach that utilizes the CES function and illustrated the applicability and usefulness of it with practices on two well-known models. We believe that the approach can be used in various other network formation setups and deliver meaningful messages.

The paperhu also offers a prospect for empirical research on network formation. The variation we introduce on link investment substitutability could be an omitted variable in works that adopt the bilateral or unilateral link formation assumption. Adding a degree of freedom along that dimension can reduce the bias of estimator significantly in certain cases. Given the wide adoption of the CES function in macro literature, we believe the practice should not be very challenging. We believe that it would be a promising research agenda to further explore the connection between link investment substitutability and network formation, especially for more general network utility functions, instead of the two particular games analysed in this paper. Such an analysis might provide us with a network formation explanation of welfare disparity among agents.

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Appendix

Proof for Remark 1

We first prove the "if" part. Let g be a pairwise stable network and let a be an investment profile where $a = g$ so that $a_{ij} = a_{ji} = 1$ when $g_{ij} = 1$ and $a_{ij} = a_{ji} = 0$ when $g_{ij} = 0$. For a network formation game with the bilateral network formation assumption, we have $g(a) = a$. To show that g is weighted pairwise stable, we only need to show that for all pairs of players $(i, j) \in N^2$: the two conditions in Definition 2 are satisfied.

For (i), if $a_{ij} = 1$, let $a' = g - ij$. Since g is pairwise stable, we have $u_i(a) \geq u_i(a')$. Since the utility of a player does not rise with unproductibe investment of other players, we have$u_i(a) \geq u_i(a') \geq u_i(a'_{ij}, a_{-ij})$. Given that $A_{ij} = \{0, 1\}$, we have shown that for all $a'_{ij} \in A_{ij}$:$u_i(a) \geq u_i(a'_{ij}, a_{-ij})$.

It can be shown in the same way that for all $a'_{ji} \in A_{ji}$: $u_j(a) \geq u_j(a'_{ji}, a_{-ji})$.

For (ii), if $a_{ij} = 0$, we know $a_{ji} = 0$ as well. Since $A_{ij} = A_{ji} = \{0, 1\}$, there are only three types of deviations i and j can make: $a_{ij'} = 1, a'_{ji} = 0$; $a_{ij'} = 0, a'_{ji} = 1$; and $a_{ij'} = a'_{ji} = 1$. The first two kinds of deviations obvioulsy cannot benefit i and j since unproductive investment from others does not benefit a player. For the third deviation, we have $(a'_{ij}, a'_{ji}, a_{-ij,ji}) = (a'_{ji}, a'_{ij}, a_{-ji,ij}) = g + ij$. Since g is pairwise stable, we know that it cannot be

$$\begin{cases} u_i(a'_{ij}, a'_{ji}, a_{-ij,ji}) = u_i(g + ij) \geq u_i(g) = u_i(a) \\ u_j(a'_{ij}, a'_{ji}, a_{-ij,ji}) = u_j(g + ij) \geq u_j(g) = u_j(a) \end{cases}$$

with at least one inequality being strict.

We now move to prove the "only if" part. Let g be weighted pairwise stable. We know there exists an $a \in A$ such that g is created by a and players do not want to deviate from a. Let $\hat{a}$ be an investment profile where $\hat{a} = g$, we can show tha $a = \hat{a}$. Suppose this is not the case, since $g = g(a)$ and $g_{ij}(a) = \min\{a_{ij}, a_{ji}\}$, we have $a \geq g = \hat{a}$. If $a \neq \hat{a}$, then there must exists $(i, j) \in N^2$ such that $a_{ij} = 1$, $a_{ji} = 0$ and $\hat{a_{ij}} = \hat{a_{ji}} = g_{ij} = 0$. Since the utility of a player is decreasing in unproductive investment of his own and unaffected by unproductibe investments of others, a deiviation where $a'_{ij} = a'_{ji} = 0$ will lead to

$$\begin{cases} u_i(a'_{ij}, a'_{ji}, a_{-ij,ji}) > u_i(a) \\ u_j(a'_{ij}, a'_{ji}, a_{-ij,ji}) = u_j(a) \end{cases}$$

contradicting condition (ii) of Definition 2.

To show that g is pairwise stable, we only need to show that for all pairs of players $(i, j) \in N^2$: the two conditions in Definition 3 are satisfied.

For (i), if $g_{ij} = 1$, since $g = \hat{a}$ and for all $a'_{ij} \in A_{ij}$: $u_i(\hat{a}) \geq u_i(a'_{ij}, \hat{a_{-ij}})$. We have$u_i(g) \geq u_i(a'_{ij}, \hat{a_{-ij}})$. Given that the utility of a player is unaffected by unproductive link investments of others,$u_i(g) \geq u_i(a'_{ij}, a'_{ji}, \hat{a_{-ij,ji}})$ where $a'_{ij} = a'_{ji} = 0$, so $u_i(g) \geq u_i(g - ij)$.

Similarily, we can show that$u_j(g) \geq u_j(g - ij)$.

For (ii), if $g_{ij} = 0$, let $a'_{ij} = a'_{ji} = 1$ so that $(a_{ij}, a'_{ji}, \hat{a_{-ij,ji}}) = g + ij$. Since $\hat{a}$ satisfies condition (ii) of Definition 2, we know that it cannot be that

$$\begin{cases} u_i(g + ij) \geq u_i(g) \\ u_j(g + ij) \geq u_j(g) \end{cases}$$

with at least one inequality being strict.

Proof for Proposition 1

We first show that the cost of a link is minimized when $a_{ij} = a_{ji} = 1$ for $\beta < 1$, $a_{ij} + a_{ji} = 2$ for $\beta = 1$, and $a_{ij} = 2^{\frac{1}{\beta}}, a_{ji} = 0$ or $a_{ij} = 0, a_{ji} = 2^{\frac{1}{\beta}}$ for $\beta > 1$. To minimize the cost of a link, we solve

$$\min(a_{ij} + a_{ji})c \quad \text{s.t.} \quad a_{ij} \geq 0, a_{ji} \geq 0 \text{ and } (\frac{1}{2}a_{ij}^{\beta} + \frac{1}{2}a_{ji}^{\beta})^{\frac{1}{\beta}} \geq 1$$

Given that the last constraint should always be binding, we solve

$$\min[a_{ij} + 2^{\frac{1}{\beta}}(1 - \frac{1}{2}a_{ij}^{\beta})^{\frac{1}{\beta}}]c \quad \text{s.t.} \quad a_{ij} \geq 0 \text{ and } 1 - \frac{1}{2}a_{ij}^{\beta} \geq 0$$

The first order derivative of the objective function w.r.t. a_{ij} is

$$1 + 2^{\frac{1}{\beta}}\frac{1}{\beta}(1 - \frac{1}{2}a_{ij}^{\beta})^{\frac{1}{\beta}-1}(-\frac{1}{2})\beta a_{ij}^{\beta-1}$$

which is equal to 0 for all a_{ij} when $\beta = 1$ and for $a_{ij} = 2^{\frac{1}{\beta}}(1 - \frac{1}{2}a_{ij}^{\beta})^{\frac{1}{\beta}} = a_{ji} = 1$ when $\beta \neq 1$.

The second order derivative of the objective function w.r.t a_{ij} is

$$\begin{aligned}
&2^{\frac{1}{\beta}-1}[(\frac{1}{\beta} - 1)(1 - \frac{1}{2}a_{ij}^{\beta})^{\frac{1}{\beta}-2}(-\frac{1}{2})\beta a_{ij}^{2\beta-2} + (1 - \frac{1}{2}a_{ij}^{\beta})^{\frac{1}{\beta}-1}(\beta - 1)a_{ij}^{\beta-2}] \\
=&2^{\frac{1}{\beta}-1}(\beta - 1)[(\frac{1}{2})(1 - \frac{1}{2}a_{ij}^{\beta})^{\frac{1}{\beta}-2}\beta a_{ij}^{2\beta-2} + (1 - \frac{1}{2}a_{ij}^{\beta})^{\frac{1}{\beta}-1}a_{ij}^{\beta-2}]
\end{aligned}$$

which is positive when $\beta > 1$ and negative when $\beta < 1$.

With the above results on first and second order derivative, we know that when $\beta = 1$, any (a_{ij}, a_{ji}) such that $(\frac{1}{2}a_{ij}^{\beta} + \frac{1}{2}a_{ji}^{\beta})^{\frac{1}{\beta}} = 1$ solves the problem of link cost minimization. That is, when $\beta = 1$, any (a_{ij}, a_{ji}) such that $a_{ij} + a_{ji} = 2$ solves the problem. When $\beta < 1$, the cost minimizing problem has an interior solution of $a_{ij} = a_{ji} = 1$, and when $\beta < 1$, the problem has boundary solutions of $a_{ij} = 2^{\frac{1}{\beta}}$ and $a_{ji} = 0$ or $a_{ij} = 0$ and $a_{ji} = 2^{\frac{1}{\beta}}$.

This shows that the cost minimizing total expenditure of a link is $2c$ when $\beta \leq 1$ and $2^{\frac{1}{\beta}}$ when $\beta > 1$. The efficient network structure characterization then follows directly from Proposition 1 of Jackson and Wolinsky (1996).

Proof for Lemma 1

For point (i), first it's obvious that for all $k \in N$, either $d_{ik}(g + ij) < d_{ik}(g)$ or $d_{ik}(g + ij) = d_{ik}(g)$ since the addition of a link would not make the distance from i to other nodes longer. If $d_{ik}(g + ij) < d_{ik}(g)$, it must be that there is a new path from i to k that is shorter than all original paths. Since there is only one such new path that travels through j, the condition for $d_{ik}(g+ij) < d_{ik}(g)$ to be the case is $d_{jk}(g)+1 < d_{ik}(g)$ and the new distance is $d_{jk}(g)+1$. Note when $d_{jk}(g)+1 = d_{ik}(g)$, we also have $d_{ik}(g+ij) = d_{jk}(g)+1$. So, when $d_{jk}(g)+1 \leq d_{ik}(g)$,, which is equivalent to $d_{jk}(g) < d_{ik}(g)$, we have $d_{ik}(g+ij) = d_{jk}(g)+1$. For other cases, $d_{ik}(g+ij) = d_{ik}(g)$.

For point (ii), we have for all $k \in N$, either $d_{ik}(g-ij) > d_{ik}(g)$ or $d_{ik}(g-ij) = d_{ik}(g)$ since the deduction of a link would not make the distance from i to other nodes shorter. If $d_{ik}(g-ij) > d_{ik}(g)$, it must be that the original shortest path from i to k that travels through j. The condition for that to be the case is $d_{ik}(g) > d_{jk}(g)$ and the new distance is $\min_{l:l\neq j, d_{il}(g=1} d_{lk}(g)+1$.

Proof for Proposition 2

For the "if" part, we need to show that if a network g satisfies the condition in Proposition 2, then we can find a investment profile $a \in A$ such that $g(a) = g$ and a satisfies the conditions in Definition 2 for g to be weighted pairwise stable.

Let a be an investment profile where $a_{ij} = a_{ji} = 0$ when $g_{ij} = 0$ and $a_{ij} = \frac{MB(i\leftarrow j,g)}{c}$ and $a_{ji} = 2^{\frac{1}{\beta}}(1-\frac{1}{2}a_{ij}^{\beta})^{\frac{1}{\beta}}$ when $g_{ij} = 1$ and $i < j$. With this way of construction, we have $g_{ij}(a) = 0$ when $g_{ij} = 0$ and $g_{ij}(a) = 1$ when $g_{ij} = 1$. So, $g(a) = g$ is satisfied.

We now verify that (i) when $g_{ij} = 1$, players i and j do not want to adjust their investments to each other and (ii) when $g_{ij} = 0$, players i and j cannt form a mutually beneficial link.

For (i), when $g_{ij} = 1$, since $a_{ij} = \frac{MB(i\leftarrow j,g)}{c}$, $a_{ji} = 2^{\frac{1}{\beta}}(1-\frac{1}{2}a_{ij}^{\beta})^{\frac{1}{\beta}}$ and

$$h(\frac{MB(i\leftarrow j,g)}{c}, \frac{MB(j\leftarrow i,g)}{c}) \geq 1$$

we have

$$\left\{\frac{MB(j\leftarrow i,g)}{c} \geq 2^{\frac{1}{\beta}}(1-\frac{1}{2}(\frac{MB(i\leftarrow j,g)}{c})^{\beta})^{\frac{1}{\beta}} = 2^{\frac{1}{\beta}}(1-\frac{1}{2}a_{ij}^{\beta})^{\frac{1}{\beta}} = a_{ji}\right.$$

Hence

$$\begin{cases} MB(i\leftarrow j,g) = ca_{ij} = MC(i\rightarrow j,g) \\ MB(j\leftarrow i,g) \geq ca_{ji} = MC(j\rightarrow i,g) \end{cases}$$

Neither i nor j would want to sever the link. Since $h(a_{ij}a_{ji}) = 1$, neigher player want to increase or decrease their investments.

For (ii), when $g_{ij} = 0$, suppose there exists a'_{ij}, a'_{ji} such that $h(a'_{ij}, a'_{ji}) \geq 1$ and

$$\begin{cases} MB(i\leftarrow j,g) \geq MC(i\rightarrow j,g) = ca'_{ij} \\ MB(j\leftarrow i,g) \geq MC(j\rightarrow i,g) = ca'_{ji} \end{cases}$$

with at least one inequality being strict, then

$$h(\frac{MB(i \leftarrow j, g)}{c}, \frac{MB(j \leftarrow i, g)}{c}) > h(a'_{ij}, a_{ji'} \geq 1$$

violating $h(\frac{MB(i \leftarrow j,g)}{c}, \frac{MB(j \leftarrow i,g)}{c}) \leq 1$. Players i and j cannot find a mutually beneficial way to form a link.

For the "only if " part, we need to show that if a network g does not satisfy the conditions in Proposition 2, then we cannot find an investment profile $a \in A$ such that $g(a) = g$ and a satisfies the conditions in Definition 2 for g to be weighted pairwise stable.

Suppose there exists $i, j \in N$ such that $g_{ij} = 1$ and

$$h(\frac{MB(i \leftarrow j, g)}{c}, \frac{MB(j \leftarrow i, g)}{c}) < 1$$

then we cannot find an a where $g_{ij}(a) = h(a_{ij}, a_{ji} = 1)$ and

$$\begin{cases} MB(i \leftarrow j, g) \geq MC(i \rightarrow j, g) = ca_{ij} \\ MB(j \leftarrow i, g) \geq MC(j \rightarrow i, g) = ca_{ji} \end{cases}$$

If we can, then

$$h(\frac{MB(i \leftarrow j, g)}{c}, \frac{MB(j \leftarrow i, g)}{c}) \geq h(a_{ij}, a_{ji}) = 1$$

violating the starting assumption.

Suppose there exists $i, j \in N$ such that $g_{ij} = 0$ and

$$h(\frac{MB(i \leftarrow j, g)}{c}, \frac{MB(j \leftarrow i, g)}{c}) > 1$$

then there exists $a'_{ij} \in A_{ij}$ and $a'_{ji} \in A_{ji}$ such that $h(a'_{ij}, a'_{ji}) \geq 1$ and

$$\begin{cases} MB(i \leftarrow j, g) \geq MC(i \rightarrow j, g) = ca'_{ij} \\ MB(j \leftarrow i, g) \geq MC(j \rightarrow i, g) = ca'_{ji} \end{cases}$$

with at least one inequality being strict.

To see this, let $a'_{ij} = h(\frac{MB(i \leftarrow j,g)}{c}$ and $a'_{ji} = 2^{\frac{1}{\beta}}(1 - \frac{1}{2}a'^{\beta}_{ij})^{\frac{1}{\beta}}$ so that $h(a'_{ij}, a'_{ji}) \geq 1$ and $MB(i \leftarrow j, g) \geq ca'_{ij}$ are automatically satisfied. Since

$$h(\frac{MB(i \leftarrow j, g)}{c}, \frac{MB(j \leftarrow i, g)}{c}) > 1$$

we have

$$\frac{MB(j \leftarrow i, g)}{c} > 2^{\frac{1}{\beta}}(1 - \frac{1}{2}(\frac{MB(i \leftarrow j, g)}{c})^{\beta})^{\frac{1}{\beta}} = 2^{\frac{1}{\beta}}(1 - \frac{1}{2}a'^{\beta}_{ij})^{\frac{1}{\beta}} = a'_{ji}$$

Players i and j benefit by deviating from a.

Proof for Proposition 3

(i) The complete network is weighted pairwise stable when players do not want to sever the link between them, which requires for any $i, j \in N$:

$$h(\frac{MB(i \leftarrow j, complete)}{c}, \frac{MB(j \leftarrow i, complete)}{c}) \geq 1$$

Since for any $i, j \in N$, if they lose their link, their distances to each other become 2 and their distances to others reamain 1, we have

$$MB(i \leftarrow j, complete) = MB(j \leftarrow i, complete) = \delta - \delta^2$$

The complete network is weighted pairwise stable if and only if

$$(\frac{1}{2}(\frac{\delta - \delta^2}{c})^\beta + \frac{1}{2}(\frac{\delta - \delta^2}{c})^\beta)^{\frac{1}{\beta}} \geq 1$$

which is equivalent to

$$c \leq \delta - \delta^2$$

So, $C^{wps}(complete; \beta, \delta) = [0, \delta - \delta^2]$. We know from Proposition 1 that $C^e(complete; \beta, \delta) = [0, \frac{1}{2}^{\frac{1}{\max\{\beta,1\}}-1}(\delta - \delta^2)]$. Since $\frac{1}{2}^{\frac{1}{\max\{\beta,1\}}-1} \geq 1$,

$$C^{wps}(complete; \beta, \delta) \subset C^e(complete; \beta, \delta)$$

Therefore,

$$T(complete; \beta, \delta) = \frac{(\frac{1}{2}^{\frac{1}{\max\{\beta,1\}}-1} - 1)(\delta - \delta^2)}{\frac{1}{2}^{\frac{1}{\max\{\beta,1\}}-1}(\delta - \delta^2)} = \begin{cases} 0 \text{ when } \beta \leq 1 \\ \frac{\frac{1}{2}^{\frac{1}{\beta}-1} - 1}{\frac{1}{2}^{\frac{1}{\beta}-1}} \text{ when } \beta > 1 \end{cases}$$

Differentiate $\frac{\frac{1}{2}^{\frac{1}{\beta}-1}-1}{\frac{1}{2}^{\frac{1}{\beta}-1}}$ w.r.t. $\frac{1}{\beta}$, it is easy to see that the first order derivative is positive.

(ii) We know from Example 1 that $C^{wps}(star; \beta, \delta) = [\delta - \delta^2, \left(\frac{1}{2}\delta^\beta + \frac{1}{2}(\delta + (n-2)\delta^2)^\beta\right)^{\frac{1}{\beta}}]$ and from Proposition 1 that $C^e(star; \beta, \delta) = [\frac{1}{2}^{\frac{1}{\max\{\beta,1\}}-1}(\delta - \delta^2), \frac{1}{2}^{\frac{1}{\max\{\beta,1\}}-1}(\delta + \frac{n-2}{n}\delta^2)]$.

When $\beta \leq 1$,

$$\frac{1}{2}^{\frac{1}{\max\{\beta,1\}}-1}(\delta - \delta^2) = \delta - \delta^2$$

and

$$\begin{aligned} \frac{1}{2}^{\frac{1}{\max\{\beta,1\}}-1}(\delta + \frac{n-2}{n}\delta^2) &= \delta + \frac{n-2}{n}\delta^2 \\ &= \frac{1}{2}\delta + \frac{1}{2}(\delta + (n-2)\delta^2) \end{aligned}$$

which we show, with the power mean inequality, is not smaller than $\left(\frac{1}{2}\delta^\beta + \frac{1}{2}(\delta + (n-2)\delta^2)^\beta\right)^{\frac{1}{\beta}}$.

So,

$$T(star; \beta, \delta) = \frac{\frac{1}{2}\delta + \frac{1}{2}(\delta + (n-2)\delta^2) - \left(\frac{1}{2}\delta^\beta + \frac{1}{2}(\delta + (n-2)\delta^2)^\beta\right)^{\frac{1}{\beta}}}{\frac{1}{2}\delta + \frac{1}{2}(\delta + (n-2)\delta^2) - (\delta - \delta^2)}$$

Again, with the power mean inequality, the numerator decreases in β. As the denominator does not depend on β, $T(star; \beta, \delta)$ decreases with β and reaches 0 when $\beta = 1$.

When $\beta > 1$, it is obvious that

$$\frac{1}{2}^{\frac{1}{\max\{\beta,1\}}-1}(\delta - \delta^2) > \delta - \delta^2$$

Regarding the relationship between $\frac{1}{2}^{\frac{1}{\max\{\beta,1\}}-1}(\delta + \frac{n-2}{n}\delta^2)$ and $\left(\frac{1}{2}\delta^\beta + \frac{1}{2}(\delta + (n-2)\delta^2)^\beta\right)^{\frac{1}{\beta}}$, we arrange the expressions and find that

$$\frac{1}{2}^{\frac{1}{\beta}-1}(\delta + \frac{n-2}{n}\delta^2) = \frac{1}{2}^{\frac{1}{\beta}}(\delta + \delta + (n-2)\delta^2)$$

which is $2^{\frac{1}{\beta}}$ times a 1-norm of vector $(\delta, \delta + (n-2)\delta^2)$, and

$$\left(\frac{1}{2}\delta^\beta + \frac{1}{2}(\delta + (n-2)\delta^2)^\beta\right)^{\frac{1}{\beta}} = \frac{1}{2}^{\frac{1}{\beta}}(\delta^\beta + (\delta + (n-2)\delta^2)^\beta)^{\frac{1}{\beta}}$$

which is $2^{\frac{1}{\beta}}$ times a β-norm of vector $(\delta, \delta + (n-2)\delta^2)$.

Since $\beta > 1$,

$$\frac{1}{2}^{\frac{1}{\max\{\beta,1\}}-1}(\delta + \frac{n-2}{n}\delta^2) > \left(\frac{1}{2}\delta^\beta + \frac{1}{2}(\delta + (n-2)\delta^2)^\beta\right)^{\frac{1}{\beta}}$$

So,

$$\begin{aligned} T(star; \beta, \delta) &= \frac{\frac{1}{2}^{\frac{1}{\beta}-1}(\delta + \frac{n-2}{n}\delta^2) - \left(\frac{1}{2}\delta^\beta + \frac{1}{2}(\delta + (n-2)\delta^2)^\beta\right)^{\frac{1}{\beta}}}{\frac{1}{2}^{\frac{1}{\beta}-1}(\delta + \frac{n-2}{n}\delta^2) - \frac{1}{2}^{\frac{1}{\beta}-1}(\delta - \delta^2)} \\ &= \frac{\delta + \delta + (n-2)\delta^2 - (\delta^\beta + (\delta + (n-2)\delta^2)^\beta)^{\frac{1}{\beta}}}{2(\delta + \frac{n-2}{n}\delta^2) - 2(\delta - \delta^2)} \end{aligned}$$

Since $(\delta^\beta + (\delta + (n-2)\delta^2)^\beta)^{\frac{1}{\beta}}$ is a β-norm of vector $(\delta, \delta + (n-2)\delta^2)$, it is decreasing in β. Hence $T(star; \beta, \delta)$ is rising in β when $\beta > 1$.

(iii) The empty network is weighted pairwise stable when players do not want to form any link between then, which requires for any $i, j \in N$:

$$h(\frac{MB(i \leftarrow j, complete)}{c}, \frac{MB(j \leftarrow i, complete)}{c}) \leq 1$$

Since for any $o, j \in N$ in the empty network, if they form a link, their distances to each other become 1 and their distances to others remain ∞, we have

$$MB(i \leftarrow j, empty) = MB(j \leftarrow i, empty) = \delta$$

The empty network is weighted pairwise stable if and only if

$$(\frac{1}{2}(\frac{\delta}{c})^{\beta} + \frac{1}{2}(\frac{\delta}{c})^{\beta})^{\frac{1}{\beta}} \leq 1$$

which is equivalent to $c \geq \delta$. So, $C^{wps}(empty; \beta, \delta) = [\delta, n]$. We know from Proposition 1 that $C^{e}(empty; \beta, \delta) = [\frac{1}{2}^{\frac{1}{\max\{\beta,1\}}-1}(\delta + \frac{n-2}{2}\delta^2), n]$. Since $\frac{1}{2}^{\frac{1}{\max\{\beta,1\}}-1}(\delta + \frac{n-2}{2}\delta^2) > \delta$ for all $\beta \in \mathbb{R}$, $T(empty; \beta, \delta) = 0$ for all $\beta \in \mathbb{R}$.

Proof of Propsition 4

We first list some basic conditions for s^* to be a Nash equilibrium.

Lemma 2. *If $s^* \in S$ is a Nash equilibrium, the for all $i \in N$: (i) $x_i^* + \sum_{j \neq i} h(a_{ij}^*, a_{ji})x_j \geq \hat{y}$, and $x_i^* + \sum_{j \neq i} h(a_{ij}^*, a_{ji})x_j = \hat{y}$ if $x_i^* > 0$, (ii) $x_i^* \leq \hat{y}$, and (iii) either $x_i^* > 0$ or $a_{ij}^* > 0$ for some j.*

Proof. For the first point, suppose $x_i^* + \sum_{j \neq i} h(a_{ij}^*, a_{ji})x_j < \hat{y}$, then since $f'(\hat{y}) = c$ and $f''() < 0$, we have $f'(x_i^* + \sum_{t \neq i} h(a_{it}^*, a_{ti})x_t) - c > 0$, player i will want to put in more effort and s_i^* cannot be a best response, indicating that s^* cannot be a Nash equilibrium. When $x_i^* > 0$, suppose $x_i^* + \sum_{j \neq i} h(a_{ij}^*, a_{ji})x_j > \hat{y}$, we can infer that $f'(x_i^* + \sum_{t \neq i} h(a_{it}^*, a_{ti})x_t) - c < 0$, player i will want to put in less effort and s_i^* cannot be a best response. Knowing that $x_i^* + \sum_{j \neq i} h(a_{ij}^*, a_{ji})x_j \geq \hat{y}$, we get $x_i^* + \sum_{j \neq i} h(a_{ij}^*, a_{ji})x_j = \hat{y}$ when $x_i^* > 0$. For the second point, since $h(a_{ij}^*, a_{ji})x_j \geq 0$ $\forall j \neq i$, together with the first point, we know $x_i^* \leq \hat{y}$. Finally, in equilibrium it cannot be that $x_i^* = 0$ and $a_{ij}^* = 0$ $\forall j \neq i$. This is because if $x_i^* = 0$, then $a_{ji}^* = 0$ $\forall j \neq i$, as linking to i incurs no benefit but cost for j. Thus $h(a_{ij}^*, a_{ji}^*) = h(0, 0) = 0$ $\forall j \neq i$. This leads to $x_i^* + \sum_{j \neq i} h(a_{ij}^*, a_{ji}^*)x_j^* = 0$, contradicting the first condition. □

We now make some notes on best responses of players. Given other players' strategy s_{-i}, player i best respond by solving a nonlinear optimization problem with the constraints that $x_i \geq 0$ and $a_{ij} \geq 0$ $\forall j \neq i$. Therefore, if $s_i^* = \{x_i^*, a_{i1}^*, ..., a_{ii-1}^*, a_{ii+1}^*, ...a_{in}^*\} \in BR_i(s_{-i})$, we have the following Karush-Kuhn-Tucker necessary conditions:

$$\begin{cases} f'(x_i^* + \sum_{l \neq i} h(a_{il}^*, a_{li})x_l) - c = 0, & \text{if } x_i^* > 0 \\ f'(x_i^* + \sum_{l \neq i} h(a_{il}^*, a_{li})x_l) - c \leq 0, & \text{if } x_i^* = 0 \end{cases} \tag{1}$$

and for all $j \neq i$:

$$\begin{cases} f'(x_i^* + \sum_{l \neq i} h(a_{il}^*, a_{li})x_l)h_x(a_{ij}^*, a_{ji})x_j - k = 0, & \text{if } a_{ij}^* > 0 \\ f'(x_i^* + \sum_{l \neq i} h(a_{il}^*, a_{li})x_l)h_x(a_{ij}^*, a_{ji})x_j - k \leq 0, & \text{if } a_{ij}^* = 0 \end{cases} \tag{2}$$

i.e. if x_i^* or a_{ij}^* is an interior solution, then the marginal benefit from it must be zero; if it is a corner solution, then the marginal benefit must be less than or equal to zero.

We know that f is a concave function and $h(a_{ij}, a_{ji})$ taken a_{ji} as given is a concave function of a_{ij} when $\beta \leq 1$. So the utility function of player i is a concave function of x_i and $\{a_{ij}\}_{j \neq i}$ when $\beta \leq 1$. Moreover we know that the inequality constraints are convex, so the KKT necessary conditions are also sufficient for optimazation when $\beta \leq 1$.

Regardless of whether the KKT conditions are sufficient, we know the following must be true for the best response:

Lemma 3. *If* $s_i^* = \{x_i^*, a_{i1}^*, ..., a_{ii-1}^*, a_{ii+1}^*, ...a_{in}^*\} \in BR_i(s_{-i})$, *then*

$$h_x(a_{ij}^*, a_{ji})x_j \begin{cases} \leq k/c & \forall j \neq i \text{ when } x_i^* > 0 \\ \geq k/c & \text{when } a_{ij}^* > 0 \\ = k/c & \text{when } x_i^* > 0 \text{ and } a_{ij}^* > 0 \end{cases}$$

Proof. First, if $x_i^* > 0$, then we know from (1) that $f'(x_i^* + \sum_{l \neq i} h(a_{il}^*, a_{li})x_l) = c$ and from (2) we know for all $j \neq i$:$f'(x_i^* + \sum_{l \neq i} h(a_{il}^*, a_{li})x_l)h_x(a_{ij}^*, a_{ji})x_j \leq k$. Combining the two relations, we have for all $j \neq i$:

$$h_x(a_{ij}^*, a_{ji})x_j \leq k/c \tag{3}$$

Then, if $a_{ij}^* > 0$, then we know from (2) that $f'(x_i^* + \sum_{l \neq i} h(a_{il}^*, a_{li})x_l)h_x(a_{ij}^*, a_{ji})x_j = k$ and from (1) we have that $f'(x_i^* + \sum_{l \neq i} h(a_{il}^*, a_{li})x_l) \leq c$. Combining the two relations, we have:

$$h_x(a_{ij}^*, a_{ji})x_j \geq k/c \tag{4}$$

Finally, if $x_i^* > 0$ and $a_{ij}^* > 0$, we know (3) and (4) are both true, so:

$$h_x(a_{ij}^*, a_{ji})x_j = k/c \tag{5}$$

□

Intuitively, we can interpret the conditions specified by in Lemma 3 as comparing the relative marginal revenue between investing in relationship with j and searching, which is $h_x(a_{ij}^*, a_{ji})x_j$, with the relative cost of the two options, which is k/c. For example, if $x_i^* > 0$ and $a_{ij}^* > 0$, it indicates that the relative marginal revenue must be equal to the relative marginal cost such that player i is indifferent between the two ways of acquiring information.

Now we start to prove Proposition 4. It is straightforward to show that if a strategy profile fits the description in Proposition 4, then it is a Nash equilibrium.: We can verify that all players are best responding. So here we only prove that a Nash equilibrium must fit the desciption in Proposition 4.

Suppose there is a Nash equilibrium s' such that there exists a resulting component $C' = (N^{C'}, g^{C'})$ that is not characterized by I, R, B, or S.

It cannot be that $\mid N^{C'} \mid = 1$. Because when $\mid N^{C'} \mid = 1$, from Lemma 2 it must be $x_i' = \hat{y}$ in equilibrium, which is the same as I.

If $\mid N^{C'} \mid > 1$, then there exisits $i, j \in N^{C'}$ such that $g_{ij}' > 0$. First look at the case when $a_{ij}' > 0$ and $a_{ji}' > 0$ for all $i, j \in N^{C'}$ where $g_{ij}' > 0$.

Since $a_{ij}' > 0$ and $a_{ji}' > 0$, we know that $x_i' > 0$ and $x_j' > 0$, otherwise at least on player would not want to invest in the link. Then from Lemma 3, we get:

$$\begin{cases} h_x(a_{ij}', a_{ji}') = \frac{k}{cx_j'} \\ h_y(a_{ij}', a_{ji}') = \frac{k}{cx_i'} \end{cases} \tag{6}$$

We also know that for CES link formation function, the following is true:

$$h_x(a_{ij}, a_{ji})^{\frac{\beta}{\beta-1}} + h_y(a_{ij}, a_{ji})^{\frac{\beta}{\beta-1}} = 2^{\frac{1}{1-\beta}}$$

Substitute $h_x(a_{ij}, a_{ji})$ and $h_y(a_{ij}, a_{ji})$ with $\frac{k}{cx_j'}$ and $\frac{k}{cx_i'}$ respectively, we get:

$$x_i'^{\frac{\beta}{1-\beta}} + x_j'^{\frac{\beta}{1-\beta}} = 2^{\frac{1}{1-\beta}}(k/c)^{\frac{\beta}{1-\beta}} \tag{7}$$

With Implict Function Theorem, We can see that $\frac{dx_j'}{dx_i'} = -(x_i'/x_j')^{\frac{2\beta-1}{1-\beta}} < 0$ for positive x_i' and x_j'. This means that if i and j are connected, then there is a one-to-one relationship between their effort.

If there exists $i, j \in N^{C'}$ s.t. $a_{ij}' > 0$ and $x_i' = x_j'$, then for all $l \in N^{C'}$, we have $x_l' = x_i' = x_j'$ given the one-to-one relationship for linked players. Then with equation (7), we get $x_l' = 2k/c$, and from equation (6), we get $h_x(a_{ij}', a_{ji}') = 1/2$, and hence $a_{ij}' = a_{ji}'$. Finally, from Lemma 2, since $x_i' + \sum_{j \neq i} h(a_{ij}', a_{ji}')x_j' = \hat{y}$, we get $\sum_{j \neq i} a_{ij}' = c\hat{y}/2k - 1$. In this case C' belongs to R type of component.

If there exists $i, j \in N^{C'}$ s.t. $a_{ij}' > 0$ and $x_i' \neq x_j'$, let $x_i' = x_1$ and $x_j' = x_2$. Since $a_{ij}' > 0$ and $a_{ji}' > 0$, from Lemma 2 we know $0 < x_i' < \hat{y}$ and $0 < x_j' < \hat{y}$, so we have $0 < x_1 < \hat{y}$ and $0 < x_2 < \hat{y}$. It is immediate to see that if there exists a sequence of nodes $i = v_0, v_1, ..., v_m = l$, such that $g_{v_{k-1}v_k} > 0$ for $k = 1, ..., m$ and m is an even number, then $x_l' = x_i' = x_1$. If there exists a sequence of nodes $i = i_0, i_1, i_2, ..., i_m = l$, such that $g_{i_k i_{k+1}} > 0$ for $k = 0, 1, ..., m-1$ and m is an odd number, then $x_l' = x_j' = x_2$. This follows from the one-to-one relationship between two connected nodes. And from this we can see for all $l \in N^{C'}$, $x_l' = x_1$ or $x_l' = x_2$.

Define $U = \{i \in N^{C'} \mid x_i' = x_1\}$ and $V = \{j \in N^{C'} \mid x_j' = x_2\}$, we can see that in equilibrium, a player from U can only be connected to a player from V and vice versa, i.e. $a_{ij}' = 0$ if $i, j \in U$ or $i, j \in V$.

Also, an examination of the CES function gives us:

$$h_x(a_{ij}, a_{ji}) = \frac{\partial h(a_{ij}, a_{ji})}{\partial a_{ij}} = (1/2)^{\frac{1}{\beta}}(1 + (a_{ji}/a_{ij})^{\beta})^{\frac{1-\beta}{\beta}} \tag{8}$$

So for $i \in U$, $j \in V$ and $g'_{ij} > 0$, from equation (8) and (7), we can infer that $a'_{ij}/a'_{ji} = (x_2/x_1)^{\frac{1}{1-\beta}}$. Finally, from Lemma 2, we can get $\sum a'_{ij} = (\frac{cx_2^\beta}{2k})^{\frac{1}{1-\beta}}(\hat{y} - x_1)$ $\forall i \in U$, and $\sum a'_{ji} = (\frac{cx_1^\beta}{2k})^{\frac{1}{1-\beta}}(\hat{y} - x_2)$ $\forall j \in V$. In this case C' belongs to B type of component.

If there exists $i, j \in N^{C'}$ such that $a'_{ij} > 0$ and $a'_{ji} = 0$, then we know that $\beta > 0$ and there exist $j \in N^{C'}$ such that $x'_j \geq 2^{\frac{1}{\beta}} k/c$. Because we have $a'_{ij} > 0$ and $a'_{ji} = 0$, if $x'_j < 2^{\frac{1}{\beta}} k/c$, then:

$$h_x(a'_{ij}, a'_{ji})x'_j = (1/2)^{\frac{1}{\beta}} x'_j < k/c$$

violating the condition for $a'_{ij} > 0$ in Lemma 3.

Since $x'_j \geq 2^{\frac{1}{\beta}} k/c$, we know that $a'_{jl} = 0$ $\forall l \neq j$. If there is a l such that $a'_{jl} > 0$, then

$$h_x(a'_{lj}, a'_{jl})x'_j > (1/2)^{\frac{1}{\beta}} 2^{\frac{1}{\beta}} k/c = k/c.$$

violating the condition for $x'_l > 0$ in Lemma 3, so $x'_l = 0$. But as $x'_l > 0$, we have $a'_{jl} = 0$.

Now, for all l such that $g'_{jl} > 0$, we know that $a'_{jl} = 0$, hence $a'_{lj} > 0$. It must be that $x'_l = 0$, because otherwise

$$h_x(a'_{jl}, a'_{lj})x'_l \to +\infty > k/c$$

violating the condition for $x'_j > 0$ in Lemma 3.

So for all l such that $g'_{jl} > 0$, $x'_l = 0$. From Lemma 2, we can infer $x'_j = \hat{y}$.

Returning to the link $g'_{ij} > 0$ we looked at in the start, since $g'_{ij} > 0$, we know that $x'_i = 0$. This implies that $a'_{li} = 0$ $\forall l \neq i$ since linking to i brings no benefit for l. So for all l such that $g'_{il} > 0$, we know that $a'_{li} = 0$, hence $a'_{il} > 0$. We can show $x'_l = \hat{y}$ using the same logic for showing $x'_j = \hat{y}$.

In brief, we have now proved that if there exists $i, j \in N^{C'}$ such that $a'_{ij} > 0$ and $a'_{ji} = 0$, then $x'_i = 0$, $x'_j = \hat{y}$. And for all players l that is connected to j, we have $a'_{jl} = 0$, $a'_{lj} > 0$, and $x'_l = 0$. For all players l connected to i, we have $a'_{li} = 0$, $a'_{il} > 0$ $x'_l = \hat{y}$.

With the above result, using iterative reasoning, it's immediate to see that for $a'_{ij} > 0$ and $a'_{ji} = 0$, if there exists a sequence of nodes $i = v_0, v_1, ..., i_m = l$, such that $g_{v_{k-1}v_k} > 0$ for $k = 1, ..., m$ and m is an even number, then $x'_l = x'_i = 0$. If there exists a sequence of nodes $i = v_0, v_1, ..., i_m = l$, such that $g_{v_{k-1}v_k} > 0$ for $k = 1, ..., m$ and m is an odd number, then $x'_l = x'_j = \hat{y}$. And from this we can see for all $l \in (N^{C'}, g^{C'})$, $x'_l = 0$ or $x'_l = \hat{y}$.

Define $U' = \{i \in N^{C'} \mid x'_i = 0\}$ and $V' = \{j \in N^{C'} \mid x'_j = \hat{y}\}$, we can see that for $i \in U'$: $a'_{ij} > 0$ only if $j \in V'$, and for $a'_{ij} > 0$, we know the following from the best response analysis:

$$f'(x'_i + \sum_{j \in V'} h(a'_{ij}, a'_{ji})x'_j)h_x(a'_{ij}, a'_{ji})x'_j - k = 0$$

Substitute the variables with our results, get:

$$f'(\sum_{j\in V'}(1/2)^{\frac{1}{\beta}}a'_{ij}\hat{y})(1/2)^{\frac{1}{\beta}}\hat{y}-k=0$$

Let $\hat{z}$ be a scaler such that $f'(\hat{z})=2^{\frac{1}{\beta}}k/\hat{y}$, we can see $\sum_{j\in V'}a'_{ij}=2^{\frac{1}{\beta}}\hat{z}/\hat{y}$.

For $j\in V'$, we have already shown that $a'_{jl}=0\ \forall l\neq j$. We can see that C' belong to S type of components now.

We have looked at all possibilities for $C'=(N^{C'},g^{C'})$ under Nash equilibrium, and show that it can always be characterized by one of the I,R,B or S components.

Now we show that equilibrium network can only be the specific kind of combinations of I,R,B and S components for different parameter ranges. When $\hat{y}\leq 2k/c$, we need to show that there does not exist $s'\neq s^*$ such that s' is a Nash equilibrium. Suppose there is a s' where $\exists i\in N$ such that $x'_i\neq\hat{y}$ or $\exists i,j\in N$ such that $a'_{ij}>0$.

If $x'_i\neq\hat{y}$, from Lemma 2 we know that it must be $\sum_{j\neq i}h(a'_{ij},a'_{ji})>0$, indicating it cannot be $a'_{ij}=a'_{ji}=0$. So for $s'\neq s^*$ to be an equilibrium, there exists $i,j\in N$ such that $a'_{ij}>0$.

When $\hat{y}<2k/c$, from Lemma 2 we know$x'_j\leq\hat{y}<2k/c$, and from Lemma 3 we know$h_x(a'_{ij},a'_{ji})x'_j\geq k/c$. Combining the above two conditions, we need:

$$h_x(a'_{ij},a'_{ji})>1/2 \tag{9}$$

An examination of how $h_x(a_{ij},a_{ji})$ varies with a_{ji}/a_{ij} tells that when $\beta<0$, (9) implies$a'_{ji}/a'_{ij}>1$. So $a'_{ji}>0$ as well. Then we can do the same analysis for a'_{ji} and get$a'_{ij}/a'_{ji}>1$. A contradiction.

When $\hat{y}=2k/c$, if both x'_i and x'_j are less than $\hat{y}$, we can again get $x'_i<2k/c$ and $x'_j<2k/c$. Using the same logic we arrive at a contradiction. If x'_i or x'_j equal to $\hat{y}$, then from Lemma 2, we have $h(a'_{ij},a'_{ji})=0$, indicating $a'_{ij}=a'_{ji}=0$. A contradiction.

So when $\hat{y}\leq 2k/c$, the unique equilibrium is s^* where $a^*_{ij}=0$ for all $i,j\in N$, and $x^*_i=\hat{y}$ for all $i\in N$.

When $\beta<\frac{ln2}{ln(c\hat{y}/k)}$, we only need to show that there cannot be any S component in equilibrium when. We prove this by showing that if s^* is a Nash equilibrium, then $\nexists i,j\in N$ such that $a^*_{ij}>0$ and $a^*_{ji}=0$. If $a^*_{ji}=0$, then for $s^*_i\in BR_i(s^*_{-i})$, it must be $a^*_{ij}=0$, otherwise:

$$h_x(a^*_{ij},a^*_{ji})x^*_j<(1/2)^{\frac{1}{\beta}}2^{\frac{1}{\beta}}k/c=k/c$$

violating the condition for $a^*_{ij}>0$ in Lemma 3. So there cannot be unilaterally sponsored link in equilibrium when $\hat{y}<2^{\frac{1}{\beta}}k/c$. But all links in S are unilarterally sponsored, so there cannot be any S component in this case.

When $\beta=\frac{ln2}{ln(c\hat{y}/k)}$, nothing to be further proved.

When $\beta>\frac{ln2}{ln(c\hat{y}/k)}$, we first show that there cannot be any I component in equilibrium. If there is an equilibrium s' with a I component, then there exist $j\in N$ such that $j\in N^I$, indicating $x'_j=\hat{y}$ and $g'_{ij}=0\ \forall i\neq j$.

Then $\forall i \neq j$ we have $x'_i = 0$, because it is always true that:

$$h_x(a'_{ij}, a'_{ji})x'_j > (1/2)^{\frac{1}{\beta}} 2^{\frac{1}{\beta}} k/c = k/c$$

violating the condition for $x'_i > 0$ in Lemma 3. Moreover, from Lemma 2 we know $x'_i + \sum_{l \neq i} g'_{il} a'_l \geq \hat{y} \; \forall i \neq j$. But as $x'_i = 0$ and $x'_l = 0 \; \forall l \neq j$, we can infer that $g'_{ij} > 0$, violating the assumption.

So when $\beta > \frac{ln2}{ln(c\hat{y}/k)}$, we can only have R, B, and S components in equilibrium. There are seven possible combinations for the three elements: (1) all components are R; (2) all components are B; (3) all components are S; (4) there are R and B components; (5) there are R and S components; (6) there are B and S components; and (7) all three kinds of components coexists.

We show that if there is an S component in equilibrium, then $\mid V' \mid = 1$ and there cannot be any other component. This is because for $j \in V'$, we have $x^*_j = \hat{y}$ and $a^*_{ji} = 0 \; \forall i \neq j$, then we can follow the same argument above to show that $x^*_i = 0$ and $g^*_{ij} > 0 \; \forall i \neq j$. This eliminates the possibility that an S coexists with other components or $\mid V' \mid > 1$.

Hence we know that combination (5) (6) and (7) cannot be equilibrium, and for situation (3), we know there can only be one component S with $\mid V' \mid = 1$.

Proof for Proposition 5

We first prove that when $\hat{y} < 2^{\frac{1}{\beta}} k/c$, the strategy profile s^* where $a^*_{ij} = 0 \; \forall i \neq j$ and $x^*_i = \hat{y} \; \forall i \in N$ is an equililbrium.

For all player $i \in N$, he is facing $x^*_j = \hat{y}$ and $a^*_{ji} = 0 \; \forall j \neq i$. Hence $h_x(a_{ij}, a^*_{ji})x^*_j = (1/2)^{\frac{1}{\beta}} x^*_j < k/c$ for all $a_{ij} > 0$. From Lemma 3, we know that if $s'_i \in BR_i(s^*_{-i})$, then $a'_{ij} = 0 = a^*_{ij} \; \forall j \neq i$. Then, from Lemma 2, we get $x'_i = \hat{y} = x^*_i$. Since $BR_i(s_{-i}) \neq \emptyset$, we know $s^*_i \in BR_i(s^*_{-i}) \; \forall i \in N$. So s^* is a Nash equililbrium.

We can also prove that there is no other Nash equilibrium when $\hat{y} < 2^{\frac{1}{\beta}} k/c$. Suppose there is a strategy profile $s' \neq s^*$ that is also an equilibrium. It can either be that there exists $i \in N$ where $x'_i \neq \hat{y}$ or there exists $i, j \in N$ where $a'_{ij} > 0$. If $x'_i \neq \hat{y}$, from Lemma 2, we need $h(a'_{ij}, a'_{ji})x'_j > 0$, so we have at least $a'_{ij} > 0$ or $a'_{ji} > 0$. Thus if there is a equilibrium $s' \neq s^*$, then there exists $i, j \in N$ where $a'_{ij} > 0$.

Since $a'_{ij} > 0$, from Lemma 3 we know $h_x(a'_{ij}, a'_{ji})x'_j \geq k/c$.

We also know that $h_x(a'_{ij}, a'_{ji}) \leq (1/2)^{\frac{1}{\beta}}$ when $\beta > 1$, so we need $x'_j \geq 2^{\frac{1}{\beta}} k/c > \hat{y}$, violating Lemma 2. So s' cannot be an equilibrium.

We have proved that when $\hat{y} < 2^{\frac{1}{\beta}} k/c$, the strategy profile s^* where he strategy profile s^* where $a^*_{ij} = 0 \; \forall i \neq j$ and $x^*_i = \hat{y} \; \forall i \in N$ is the unique Nash equilibrium.

When $\hat{y} \geq 2^{\frac{1}{\beta}} k/c$, showing that a single S component with $\mid V' \mid = 1$ is a Nash equilibrium is a matter of checking whether all players are best responding, which is straight-forward. Here, we focus on demonstrating the necessary conditions of a Nash equilibrium.

First, it is easy to see from Lemma 2 that $x_i^* \in [0, \hat{y}]$ for all $i \in N$. We show that there can be at most one player searching for more than $2^{\frac{1}{\beta}} \frac{k}{c}$. Suppose there are two players $i, j \in N$ with $x_i^* > 2^{\frac{1}{\beta}} \frac{k}{c}$ and $x_j^* > 2^{\frac{1}{\beta}} \frac{k}{c}$.

When $a_{ij}^* > 0$ and $a_{ji}^* > 0$, player i will find it utility enhancing to cut his effort to 0 and get x_i^* amount of information by investing more to the relationship with j. To see this, if s^* with $x_i^* > 2^{\frac{1}{\beta}} \frac{k}{c}$, $x_j^* > 2^{\frac{1}{\beta}} \frac{k}{c}$, $a_{ij}^* > 0$ and $a_{ji}^* > 0$ is a Nash equilibrium, then from Lemma 3, $h_x(a_{ij}^*, a_{ji}^*)x_j^* = \frac{k}{c}$. An investmet to j with intensity $a'_{ij} > a_{ij}^*$ will lead to $h_x(a'_{ij}, a_{ji}^*)x_j^* > h_x(a_{ij}^*, a_{ji}^*)x_j^*$ given that $\beta > 1$, and hence $h_x(a'_{ij}, a_{ji}^*)x_j^* > \frac{k}{c}$. Investing to j is a more cost efficient way than searching on his own for player i.

When not both a_{ij}^* and a_{ji}^* greater than zero, without loss of generality, let $a_{ji}^* = 0$, then for any $a_{ij} \in A_{ij}$: $h_x(a_{ij}, a_{ji}^*)x_j^* = \frac{1}{2}^{\frac{1}{\beta}}$ and hence $h_x(a_{ij}, a_{ji}^*)x_j^* > \frac{1}{2}^{\frac{1}{\beta}} 2^{\frac{1}{\beta}} \frac{k}{c} = \frac{k}{c}$. Investing to j is a more cost efficient way than searching on his own for player i. So s^* with $x_i^* > 2^{\frac{1}{\beta}} \frac{k}{c}$ cannot be a Nash equilibrium.

We now move to the second point of Proposition 5 when $\hat{y} \geq 2^{\frac{1}{\beta}} \frac{k}{c}$.

First, suppose there exsist two players $i, j \in N$ with $a_{ij}^* > 0$ and $a_{ji}^* > 0$ in a Nash equilibrium s^*. This implies $x_i^* > 0$ and $x_j^* > 0$ as otherwise linking incurs no benefit and one of the players would find it optimal to withdraw the link investments. Since $x_i^* > 0$ and $a_{ij}^* > 0$, from Lemma 3, we have $h_x(a_{ij}^*, a_{ji}^*)x_j^* = \frac{k}{c}$.

Let $a'_{ij} > a_{ij}^*$, given $\beta > 1$, we have $h_x(a'_{ij}, a_{ji}^*)x_j^* > h_x(a_{ij}^*, a_{ji}^*)x_j^* = \frac{k}{c}$. Investing to j is a more cost efficient way than searching on his own for player i. We must have $x_i^* = 0$. This violates the previous implication of $x_i^* > 0$.

Second, we show that only players who search for at least $2^{\frac{1}{\beta}}$ receive link investments. When $\beta \geq 1$, the CES function has the feature that for all $a_{ij}, a_{ji} \geq 0$: $h_x(a_{ij}, a_{ji}) \leq \frac{1}{2}^{\frac{1}{\beta}}$. Hence $h_x(a_{ij}, a_{ji})x_j < \frac{1}{2}^{\frac{1}{\beta}} 2^{\frac{1}{\beta}} \frac{k}{c} = \frac{k}{c}$ if $x_j < 2^{\frac{1}{\beta}} \frac{k}{c}$.

From Lemma 3, we know that for any $i, j \in N$, i cannot be best responding by having $a_{ij}^* > 0$ when $x_j^* < 2^{\frac{1}{\beta}} \frac{k}{c}$.

Proof for Equilibrium features when $\beta = 1$

We show that when $\beta = 1$, if $\hat{y} < 2^{\frac{1}{\beta}} k/c$, the game has a unique Nash equilibrium where each player forms an I component. If $\hat{y} \geq 2^{\frac{1}{\beta}} k/c$, then a single S component with $\mid V' \mid = 1$ is always a Nash equilibrium.

A Nash equilibrium s^* when $\beta = 1$ and $\hat{y} \geq 2^{\frac{1}{\beta}} k/c$ must satisfy (i) the effort level of all players lies between $[0, \hat{y}]$ and there can be at most one player who makes for more than $2^{\frac{1}{\beta}} k/c$ amount of effort, and (ii) only players who search for at least $2^{\frac{1}{\beta}} k/c$ receive link investments.

The results when $\hat{y} < 2^{\frac{1}{\beta}}$ can be proved in the same way as in the proof for

Proposition 5.

When $\hat{y} < 2^{\frac{1}{\beta}}$, it is straight-forward to verify that a single S component with $\mid V' \mid= 1$ is a Nash equilibrium. We again focus on demonstrating the necessary conditions of a Nash equilibrium.

Since $\beta = 1$, we have for all $a_{ij} \in A_{ij}$, $a_{ji} \in A_{ji}$: $h_x(a_{ij}, a_{ji}) = \frac{1}{2}$.

For condition (i), Lemma 2 shows that the effort level of all players lies between $[0, \hat{y}]$. If player j makes more than $2^{\frac{1}{\beta}} k/c = 2\frac{k}{c}$ amount of effort, then for all $i \neq j$, $h_x(a_{ij}, a_{ji})x_j > \frac{k}{c}$. Investing to j is a more cost efficient way than searching on his own for player i. So if s^* is a Nash equilibriun and $x_j^* > 2^{\frac{1}{\beta}} k/c$, it must be that for all $i \neq j$, $x_i^* = 0$.

For condition (ii), for all players $j \in N$, if $x_j < 2^{\frac{1}{\beta}} \frac{k}{c} = 2\frac{k}{c}$, we know $h_x(a_{ij}, a_{ji})x_j < \frac{1}{2} 2\frac{k}{c} = \frac{k}{c}$. From Lemma 3, we know that for any $i \in N$, he cannot be best responding by having $a_{ij}^* > 0$ when $x_j^* < 2^{\frac{1}{\beta}} \frac{k}{c}$.

Proof for Proposition 6

Following our equilibrium characterization in Propositions 4 and 5, we know that a single S component with $\mid V' \mid= 1$ is never a Nash equilibrium when $\beta < \frac{ln2}{ln(c\hat{y}/k)}$ but always a Nash equilibrium when $\beta \geq \frac{ln2}{ln(c\hat{y}/k)}$. Hence, bothe $SD_x(\beta, f, k, c)$ and $SD_a(\beta, f, k, c)$ are less than 1 when $\beta < \frac{ln2}{ln(c\hat{y}/k)}$ and equal to 1 when $\beta \geq \frac{ln2}{ln(c\hat{y}/k)}$.

We now show that they both rise with β when $\beta < \frac{ln2}{ln(c\hat{y}/k)}$.

Let x_{sup}^* and x_{inf}^* be the supremum and infimum amount of effort a player makes in equilibrium respectively. From Proposition 4, we know that when $\beta < \frac{ln2}{ln(c\hat{y}/k)}$, $x_{sup}^* = \hat{y}$ and

$$x_{inf}^{*\,\frac{\beta}{1-\beta}} + \hat{y}^{\frac{\beta}{1-\beta}} = 2^{\frac{1}{1-\beta}} \frac{k}{c}^{\frac{\beta}{1-\beta}} \tag{10}$$

We know that

$$SD_x(\beta, f, k, c) = \frac{x_{sup}^* - x_{inf}^*}{x_{sup}^* + x_{inf}^*} = \frac{\hat{y} - x_{inf}^*}{\hat{y} + x_{inf}^*}$$

which is decreasing in x_{inf}^*.

So, to show $SD_x(\beta, f, k, c)$ is increasing in β, we only need to show that x_{inf}^* is decreasing in β. Going back to the relationship of equilibrium between connected players:

$$x_1^{\frac{\beta}{1-\beta}} + x_2^{\frac{\beta}{1-\beta}} = 2^{\frac{1}{1-\beta}} \frac{k}{c}^{\frac{\beta}{1-\beta}}$$

We can see that there is a negative relationship between x_1 and x_2, we can also check how fast x_1 decrease with x_2:

$$\frac{d\frac{dx_1}{dx_2}}{d\beta} = -(x_2/x_1)^{\frac{2\beta-1}{1-\beta}} \frac{1}{(1-\beta)^2} ln\frac{x_2}{x_1}$$

which is negative when $x_1 < x_2$. Since when $x_2 = 2k/c$, we have $x_1 = x_2$, so $\frac{d\frac{dx_1}{dx_2}}{d\beta}$ is negative when $x_2 > 2k/c$. Hence we know that for a greater β, x_1 is decreasing at a faster rate as x_2 increases when $x_2 > 2k/c$.

Let ϕ be a function such that $x_1 = \phi(x_2, \beta)$. This indicates that if $\beta_1 > \beta_2$, we have:

$$\phi(\hat{y}, \beta_1) - \phi(2k/c, \beta_1) < \phi(\hat{y}, \beta_2) - \phi(2k/c, \beta_2)$$

Since $\phi(2k/c, \beta_1) = \phi(2k/c, \beta_2) = 2k/c$, we have $\phi(\hat{y}, \beta_1) < \phi(\hat{y}, \beta_2)$ for $\beta_1 > \beta_2$, i.e. x^*_{inf} is smaller when β is greater.

For $SD_a(\beta, f, k, c)$, since $a^*_{ij}/a^*_{ji} = (x^*_j/x^*_i)^{\frac{1}{1-\beta}}$, we have

$$SD_a(\beta, f, k, c) = \sup_{s^* \in S^*} \max_{i,j \in N, g^*_{ij} > 0} \mid \frac{a^*_{ij} - a^*_{ji}}{a^*_{ij} + a^*_{ji}} \mid = 1 - \frac{2}{(\hat{y}/x^*_{inf})^{\frac{1}{1-\beta}} + 1}$$

since the player exerting most effort can always be connected to the player exerting least effort. We already proved $\frac{d(\hat{y}/x^*_{inf})}{d\beta} > 0$ just now. Also we have $\hat{y}/x^*_{inf} > 1$ and $\frac{d\frac{1}{1-\beta}}{d\beta} > 0$, hence we have $\frac{dSD_a(\beta,f,k,c)}{d\beta} > 0$ when $\beta < \frac{ln2}{ln(c\hat{y}/k)}$.

Regarding $ID_x(\beta, f, k, c)$. When $\beta < 1$, we know the regular component can always emerge in a Nash equilibrium. Since in the regular component, all players make the same amount of effort, $ID_x(\beta, f, k, c) = 0$ when $\beta < 1$. When $\beta \geq 1$, we know a single S component with $\mid V' \mid= 1$ can always emerge in equilibrium. Since in the S component, all players in U' make the same amount of effort, $ID_x(\beta, f, k, c) = 0$ when $\beta \geq 1$.

Finally, for $ID_a(\beta, f, k, c)$. We know that when $\beta \leq 1$, links can always be reciprocally sponsored, so $ID_a(\beta, f, k, c) = 0$. When $\beta > 1$, links can only be unilaterally sponsored, so $ID_a(\beta, f, k, c) = 1$.

Proof for Proposition 7

When $\beta < 1$, according to Proposition 4, for two players i and j, if $0 < x^*_i < x^*_j < \beta\hat{y}$ or $\hat{y} > x^*_i > x^*_j > \beta\hat{y}$, we know that $i, j \in N^R \cup N^B$. We also know from Lemma 2 that players in $N^R \cup N^B$ gather exactly exactly $\hat{y}$ information. So the difference in utility between i and j results from different costs they pay.

For an player i, if $i \in N^R$, we can view him as a special case of $i \in N^B$ with $x_1 = x_2 = 2k/c$. So, if $i \in N^R \cup N^B$, let $x^*_i = x_1$, we know all players connected to i exert effort level x_2 where

$$x_1^{\frac{\beta}{1-\beta}} + x_2^{\frac{\beta}{1-\beta}} = 2^{\frac{1}{1-\beta}}(k/c)^{\frac{\beta}{1-\beta}}$$

We can show that aggregate cost player i needs to pay in this case is

$$\begin{aligned} c_i(x_1) &= cx_1 + k(\frac{cx_2^\beta}{2k})^{\frac{1}{1-\beta}}(\hat{y} - x_1) \\ &= c\hat{y} - [c\hat{y} - cx_1 - k(\frac{cx_2^\beta}{2k})^{\frac{1}{1-\beta}}(\hat{y} - x_1)] \\ &= c\hat{y} - (\hat{y} - x_1)(c - k(\frac{c}{2k})^{\frac{1}{1-\beta}}(2^{\frac{1}{1-\beta}}(\frac{k}{c})^{\frac{1}{1-\beta}} - x_1^{\frac{\beta}{1-\beta}})) \\ &= c\hat{y} - (\hat{y} - x_1)k(\frac{c}{2k})^{\frac{1}{1-\beta}}x_1^{\frac{\beta}{1-\beta}} \end{aligned}$$

We can differenciate the cost w.r.t x_i and get

$$c_i'(x_1) = -k(\frac{c}{2k})^{\frac{1}{1-\beta}}[\frac{\hat{y} - x_1}{x_1}\frac{\beta}{1-\beta} - 1]$$

When $\beta > 1$, we know from Proposition 5 that all links will be unilaterally sponsored in equilibrium.

For two players $i, j \in N$ with $x_i^*, x_j^* < 2^{\frac{1}{\beta}}\frac{k}{c}$, we know from Proposition 5 that they do not receive link investments from other players, hence their utilities are determined completely by their own strategies. If their utilities are not the same, the one who receives smaller utility will want to deviate to the strategy of the other. So they must receive the same utility in equilibrium.

For two players $i, j \in N$ with $x_i^* < 2^{\frac{1}{\beta}}\frac{k}{c}$ and $x_j^* = 2^{\frac{1}{\beta}}\frac{k}{c}$, following the above reasoning, i does not receive any link investments in equilibrium but j might. If i has a higher utility than j, then j can switch his strategy to match that of i and get a higher utility.

Finally, if a player $j \in N$ makes more than $2^{\frac{1}{\beta}}\frac{k}{c}$ amount of effort, then all players that unilaterally sponsor links to j makes no effort. This is because for all $i \neq j$ with $a_{ji}^* = 0$: $h_x(a_{ij}, a_{ji}^*)x_j^* > \frac{k}{c}$. Player i best respond with $x_i^* = 0$.

Therefore, the cost for j to obtain one unit of infomation must be c.

For other players, if $i \neq j$ and $x_i^* < 2^{\frac{1}{\beta}}\frac{k}{c}$, i must be unilaterally sponsoring links to j. This is because i does not receive link investments as shown in Proposition 4 and so $h_x(a_{ij}, a_{ji}^*)x_j^* > \frac{k}{c}$. Player i best respond with $x_i^* = 0$. As j makes the most effort among all players (Proposition 5 tells us that at most one player makes more than $2^{\frac{1}{\beta}}\frac{k}{c}$ amount of effort), i best respond by unilaterally linking to j. The cost for i to obtain one unit of infomation is $\frac{c}{h_x(a_{ij}, a_{ji}^*)x_j^*} < k$, which is less than the cost for j to obtain one unit of information. Player i must be better off than j.

For other players, if $i \neq j$ and $x_i^* = 2^{\frac{1}{\beta}}\frac{k}{c}$, i must be receiving unilateral investments from j. Otherwise, as shown above, i will find unilaterally linking to j which costs only $\frac{c}{h_x(a_{ij}, a_{ji}^*)x_j^*} < k$ the most cost effective way to obtain information and does not exert effort on his own. This means that i receives free information, and the other information he obtains can not cost more than c per unit, as otherwise he will just search on his own. This makes i better off than j.