Connecting Friends

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Abstract

I investigate introductions as a way of network formation. In the model, players are endowed with different ability levels and want to match with a neighbour as capable as possible. Introductions for unacquainted neighbours influence the matching outcome since the network structure is changed. I show that a player is always willing to introduce two neighbours when at least one of them is less capable than him. If the two neighbours are both more capable, the introducer checks if there is an alternating path from one of the neighbours to him. The existence of the path is necessary for the introducer to be affected and the parity of the path length determines the direction of the effect. I also study an extension where players only have incomplete local information about the network. Now, decision-making becomes simpler. An introduction benefits the introducer when he is more capable than his current match.

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1 Introduction

The novel *Pride and Prejudice* starts with the following scene. Mr Bingley, a single man of a large fortune, just bought a house in the neighbourhood where the father, mother, and five daughters of the Bennet family live. Mrs Bennet is eager to make Mr Binley meet her daughters and marry one of them, but it seems that she lacks an excuse to visit him. Learning her concern, Elizabeth, the second daughter, says, "But you forget, mama, that we shall meet him at the assemblies, and that Mrs Long has promised to introduce him." Mrs Bennet replies, " I do not believe Mrs Long will do any such thing. She has two nieces of her own."(Austen, 1813)

This scene shows two interesting aspects of social life. First, in some scenarios, one cannot directly approach another person and propose a relationship. A pretext, such as an introduction, is needed. Second, people would reason strategically when deciding whether or not to make an introduction.

Besides the above example, there is plenty of other evidence showing the importance of introductions as a channel of network formation. A scholar can refer one young researcher to another for a joint project and a businessman can arrange meetings for two partners he has who do not know each other.[1,2] There is a much more widely documented and studied economic activity that is close to introductions in nature: referrals. Firms recruit a considerable amount of new employees through referrals of existing employees and professionals refer clients to each other when they cannot meet the demand of clients themselves.[3]

Note also that several network formation studies build upon the assumption that individuals form some of their links by meeting friends of friends.[4] This meeting process, either facilitated deliberately or unintentionally by the mutual friend, would not occur if the mutual friend is not willing to introduce his two neighbours.

Despite the importance of introductions, few studies look at the strategic reasoning behind such behaviour.[5] Most existing literature on network formation analyses the creation, maintaining, and deletion of a link by examining the incentives of players sitting on the two sides of the link. The two papers that pioneer research in the field, Jackson and Wolinsky (1996) and Bala and Goyal (2000), assume that a link is created with the mutual consent of two players and that the proposal from one player is sufficient for the formation of a link, respectively. Following studies normally adopt one of these two specifications.[6] The influence a third party can have on a relationship-facilitating

[1] Newman (2001) finds that the probability of two researchers working jointly increases with the number of coauthors they previously have in common and suggests that this is due to introductions made by the common coauthors. Zhou et al. (2003) show in their study of contractual relationships among Chinese firms, that about 40% of firms in their sample have ever formed their business connections with introductions by existing partners.

[2] Returning to introductions in the marriage market, Thornton et al. (1994) show that, in the 1980s, around 38% couples in Taiwan met each other via the introduction by a mutual friend.

[3] For a summary of the empirical literature on job referrals, see Ioannides and Datcher Loury (2004). Spurr (1988, 1990) investigates the incentive for lawyers to refer their clients to other lawyers.

[4] For example, see Granovetter (1973), Watts (2004), and Jackson and Rogers (2007).

[5] Siedlarek (2020), as far as I know, is the only other paper that studies introductions.

[6] For an overview of network formation research, see Goyal (2007), Jackson (2010), and the more

a link with an introduction, for example-has not been taken into account.

In light of these observations, I study introductions in this paper, aiming to provide a first understanding of how players choose when to facilitate an introduction of two others, and how introductions affect network stability and efficiency.

I study the following model that considers a set of players endowed with different levels of ability and are connected in a network. The players undergo a one-to-one matching process where everyone tries to find a partner among his neighbours. Here, a high-ability partner is preferred to a low-ability one. Also, having a partner is preferred to not having one. It can be shown that, given any strict ability ranking and network structure, the above process admits a unique stable matching. The preference of a player for networks is determined by how good his partner is in the stable matching under the network. I then ask: Would a player want to change the status quo network structure by introducing two of his neighbours?

I answer the question with two findings. First, I show that the matching of an introducer does not change if at least one of the introduced players has a lower ability than the introducer. Second, when both players being introduced are more capable than the introducer, the matching of the introducer will be affected only if there is an alternating path with a descending ability pattern (a path where (i) the pairs of players on it are alternatively matched and not matched with each other and (ii) the kth player on the path is more capable than the $k+2$th player for all k) starting from one of the introduced players to the introducer. The parity of the alternating path length determines the direction of the influence: an even one leads to an improvement, while an odd one leads to an impairment.

The first finding implies that a player can be very generous to his weak neighbours. This is no surprise as the weak neighbour can never successfully 'steal' a potential partner that the introducer wants to match with himself.

The second finding is more interesting. It demonstrates that an introducer can not only lose but also gain from the introduction for two more capable neighbours. It is easy to see why the introducer can be hurt by such an introduction. By introducing the pair, he is, to some extent, giving up on two potential partners: both introduced players may prefer each other more than the introducer, leaving the introducer alone. The way an introducer benefits from an introduction is indirect. Although he can lose two potential matches with the introduction, the two neighbours, if matched, free their original partners whom the introducer might prefer over his current partner and can now match with. Note that this reasoning can carry over to further players. A player who has obtained a new match with a freed-up player frees another player who will form his own new match. Observing that the effect of an introduction can pass down through a chain of new matches and freeing ups, I arrive at the alternating path characterization in the second finding. The decreasing ability pattern on the path is necessary for players to prefer their new matches to the original ones. Whether the introducer is better or worse off with the introduction is determined by his position on the path. If the path length between the introduced player and him is even, then he obtains a freed-up player; if the path length between the introduced player and him is

recent Mauleon and Vannetelbosch (2016).

odd, then he is one abandoned by his original match.

With the two findings presented above, I characterize networks that are introduction robust (networks where no player wants to introduce any pairs of his neighbours). I also characterize Pareto efficient networks and show that introduction robustness and Pareto efficiency do not imply each other.

Returning to the model setup, the baseline model of introductions has two key ingredients. First, players are equipped with heterogeneous characteristics and have aligned preferences over them. This is a realistic assumption as people differ in their education level, wealth, and physical strength and their preference for these features are aligned in most cases. Second, players do not interact with all neighbours in the network, but just one.[7] This assumption captures the fact that an agent has limited attention and resources to spare to his connections and hence, when the value a player provides to his neighbours is rivalrous, e.g., a monetary favour or a collaboration opportunity, agents need to compete for interactions with quality neighbours.

There are a few economic and social scenarios that feature this kind of competition. For example, coauthorship prevails in the academic world. Not all researchers are of the same characteristics and surely want to work with those who are smarter, easier to communicate with, and more responsible. But the number of projects a researcher can take up is limited, so a good collaborator is a blessing. A similar situation arises when starting entrepreneurs are looking for co-founders of a business. Some of the entrepreneurs must be viewed as better than others due to their stronger technical or business background. As with the earlier case, they cannot engage in multiple projects at one time because establishing a new firm requires devotion. Thus, a capable partner is a scarce resource. Further parallels can be drawn to the pursuits for attention and popularity that are common among adolescents. Adolescents can vary in their social standing and often want to have an exclusive best friend with someone popular. This competition for celebrity status is often seen in high schools, and commonly portrayed in TV series, such as *Gossip Girl*. Additionally, while it is not rare for people to have several best friends, in most cultures, individuals have only one spouse. As there are some general traits that most of the population value, e.g., beauty, and people are born with different endowments, the pursuit of an appealing partner is usually not easy. In all of the situations above, if an agent needs to form a match himself or to obtain certain scare resources from his connections, then he should think strategically before introducing connections, or, in a less direct manner, he should reason before hosting an interaction where his unacquainted friends can meet. The results in this paper shed light on the reasoning behind such decisions.

I also study two extensions of the baseline model, one where players go through a different matching process and another where players only have incomplete information about the network they are in.

For the first extension, I look at two other matching contexts. In both scenarios, players are divided into two groups and a match can only be formed by players from different groups. In one scenario, a player can only be matched once, as in our

[7] I allow players to match with more than one neighbours in the extension. As long as there is a constraint on the number of matches, the results do not change qualitatively.

baseline model. This setup is suited for the analysis of the marriage market. In the other scenario, a player can be matched to multiple players in the other group. This captures matching between managers and assistants to work on projects, upstream and downstream firms, sellers and buyers, etc. I show that the results from the baseline model only need to be adjusted slightly for the two two-sided matching settings. More specifically, first, instead of comparing the abilities of the two neighbours with his own ability, a player now compares the ability of the neighbour in his own group with his own and the ability of the neighbour in the other group with that of his partner. If one of the neighbours is less capable, the introduction would not affect the matching of the introducer. Second, a player still needs to sit on an alternating path from one of the introduced players to him for him to be affected. However, due to the two-sided nature of the matching process, an alternating path starting from the own-group neighbour always makes the introducer better off and a path starting from the other group always makes him worse off.

The robustness of my results for various matching processes shows that the understanding obtained can be applied to more general setups where links in a network provide value to players, but due to the possibly rivalrous nature of resources that one can offer to his neighbours, players need to be qualified to enjoy the benefits of social connections.

The second extension examines how players decide whether to introduce when they believe that the network is randomly formed where two players are more likely to have a link if they have similar ability levels and players only know the abilities of the neighbours they consider introducing, their partners and themselves. In this case, decision making is simple. First, an introduction that involves a neighbour who is less capable than the introducer is still harmless. Second, when considering whether to introduce two more-capable neighbours that are not one's current partner, the introducer compares his ability with his partner's ability: he is more likely to benefit rather than to lose from the introduction if he is more capable. These results for incomplete information extension show that introductions can be a common practice in social lives even though players compete for resources gained from neighbours. It is strategic for a player to always act kindly towards weaker neighbours by introducing them to others and to do favours for stronger neighbours when he is under-matched to a partner weaker than him.

This paper contributes to the literature on network formation. It is among the first that discuss how players strategically influence the links of their neighbours. Siedlarek (2020) also studies introductions. In his model, players obtain value from direct and indirect connections, as in the connections model by Jackson and Wolinsky (1996). In addition, a player earns intermediation rent if he is essential for (i.e., is the only one in between) an indirect connection. The players trade off introduction premiums he earns with a possible loss of intermediation rents when deciding whether to introduce. The author uses this framework to provide an explanation of the small-world feature we often observe in social networks. However, my model has a different utility setup and studies introduction incentives for very different scenarios. Moreover, I do not rely on the introduction premium to incentivise introductions.

This paper is also relevant to studies on job referrals. In the seminal work of Montgomery (1991), it is assumed that a worker voluntarily makes referrals for his neighbours to the firm he works at. In real life, this may not be true as workers often need to compete with co-workers for resources and promotions, and hence, choose to act strategically when referring neighbours. This paper provides a reference for how workers might reason and make strategic referrals.

The remainder of this paper is structured as follows. Section 2 explains the model setup. Section 3 illustrates the results of the baseline model. Section 4 explores extensions. Section 5 concludes. Proofs not given in the main text can be found in the Appendix.

2 The Baseline Model

2.1 Player Ability and Network

Consider a set of players $N = \{1, 2, ..., n\}$, each endowed with an ability level $a_i \in \mathbb{R}_+$ for all $i \in N$. The endowments of all players are denoted with a vector $a = (a_1, a_2, ..., a_n)$. For simplicity, I assume that there is a strict ordering of ability levels: $a_i \neq a_j$ for all $i \in N, j \neq i$.

The players are situated in a network $g = (g_{ij})_{i,j \in N}$. The variable $g_{ij} \in \{0, 1\}$ represents an undirected relationship between i and j: $g_{ij} = 1$ means there is a link between i and j, while $g_{ij} = 0$ means there is none.[8] Since relationships are undirected, g_{ij} and g_{ji} take the same value for all $i, j \in N$. I use G to denote the set of all possible networks for the n players.

I now define a few notions based on a network g. First, player i and player j are neighbours to each other if $g_{ij} = 1$. I use $N_i(g) = \{j \in N : g_{ij} = 1\}$ to denote the set of neighbours i has in g. Second, a path in a network g is a sequence of distinct players $\{i_0, i_2, ..., i_k\}$ where every two consecutive players have a link between them. Finally, if $g_{ij} = 0$, I use $g + ij$ to represent the network that is otherwise the same as g but with an additional link between i and j.

2.2 A Matching Process

Given an ability endowment a and a network structure g, players undergo a one-to-one matching process where each player partners with one of his neighbours or remains single. The outcome of the process is a matching $\mu_g : N \to N$ where $\mu_g(i) = j$ if and only if $j \in N_i(g) \cup \{i\}$ and $\mu_g(j) = i$. If $\mu_g(i) = i$, player i remains single in the matching μ_g.

I assume that players have the following aligned preference for matching. First, they always prefer to be matched to a partner with higher ability. Second, they always prefer to be matched than to remain single. Formally, let $\succ_m$ be the preference relation of player m over partners where $i \succ_m j$ means m prefers i to j as his partner, I assume

[8] As commonly specified in the literature, I let $g_{ii} = 0$ for all $i \in N$.

that for all $m \in N$ and $i, j \neq m$:

$$i \succ_m j \text{ if and only if } a_i > a_j,$$

and

$$i \succ_m m.$$

A matching under a network is *stable* if (i) no player wants to leave his current partner and stay single, and (ii) no pair of linked players want to form a new match (and leave their current partners if they have one). Formally,

Definition 1. A matching μ_g is *stable* under a network g if
(i) it is not blocked by any player i: $\nexists i \in N$ with

$$i \succ_i \mu_g(i),$$

(ii) it is not blocked by any pair of linked players (i, j): $\nexists (i, j) \in N^2$ with

$$\begin{aligned} j &\succ_i \mu_g(i), \\ i &\succ_j \mu_g(j), \text{ and} \\ j &\in N_i(g). \end{aligned}$$

Note that with the assumption that players always prefer to be matched, the first condition of a stable matching is satisfied for all possible matchings that can take place in a network. Moreover, since the preference of players depends only on the ability of others, the condition for a matching μ_g to be stable can be reformulated as there cannot exist $(i, j) \in N^2$ with

$$\begin{aligned} a_j &> a_{\mu_g(i)}, \\ a_i &> a_{\mu_g(j)}, \text{ and} \\ j &\in N_i(g). \end{aligned}$$

Due to the strictly aligned preference assumption, it is easy to see that the matching process admits a unique stable matching μ_g^* for any network $g \in G$. The stable matching can be derived with a simple algorithm where we sequentially match the highest ability player remaining with his most capable neighbour and then delete the pair from the process. This result is summarized in Lemma 1.

Lemma 1. *For all $g \in G$, the matching process admits a unique stable matching μ_g^* which can be derived with the following algorithm:*

(i) for a given network, let the player with the highest ability be matched with his most capable neighbour,

(ii) delete the pair of players who get matched in step (i) and their links from the original network and arrive at a reduced network,

(iii) go back to step (i) until the remaining network is empty or consists only of players with no links, and

(iv) all those who get matched in step (i) will be matched in μ_g^*, and all of the other players will remain single in μ_g^*.

2.3 Introduction Robust Networks and Pareto Efficient Networks

Given the unique stable matching, we can define a preference over networks for each player. A player m prefers a network g to g' if he is more satisfied with his stable matching partner under g than under g'. Specifically, let $\geq_m$ be the preference relation of player m over networks, I assume that for all $m \in N$ and $g, g' \in G$:

$$g =_m g' \text{ if and only if } \mu^*_g(m) = \mu^*_{g'}(m),$$

and

$$g >_m g' \text{ if and only if } \mu^*_g(m) \succ_m \mu^*_{g'}(m).$$

I examine the incentives of players regarding introductions based on this preference. Given a network g, a player $m \in N$ who has two unlinked neighbours $i, j \in N_m(g)$ with $g_{ij} = 0$ can introduce i and j and change the network structure from g to $g + ij$. If $g + ij >_m g$, player m wants to make the introduction. If $g >_m g + ij$, player m does not want to introduce the pair. In the tie-breaking case, we assume that m would make the introduction if he is *generous*.

A network is *introduction robust* if no player wants to make an introduction for any pair of his unlinked neighbours.

Definition 2. A network $g \in G$ is introduction robust if

(i) there does not exist $m \in N$ and $i, j \in N_m(g)$ with $g_{ij} = 0$ such that $g + ij >_m g$, and

(ii) if players are generous, there does not exist $m \in N$ and $i, j \in N_m(g)$ with $g_{ij} = 0$ such that $g + ij =_m g$.

I also look at the Pareto efficiency of networks. A network g is *Pareto efficient* if there does not exist another network g' where all players are at least as satisfied under g' as under g and some players strictly prefer g'.

Definition 3. A network $g \in G$ is Pareto efficient if there does not exist $g' \in G$ such that $g' \geq_i g$ for all $i \in N$ and that $g' >_i g$ for some $i \in N$.

3 Generosity Towards the Weak and Strategic Introductions for the Strong

In this section, I show that a generous player is always willing to introduce two neighbours when at least one of them is less capable than him as such introductions never affect the matching of the introducer. Nevertheless, when faced with a choice of whether to make two more capable neighbours meet, a player needs to judge based on his network position and the ability ranking. The key network feature a player looks for when making this decision is an *alternating path with descending ability pattern* from one of the neighbours being introduced to him. The existence of such a path is necessary for the player to be influenced by an introduction and the length of the path determines the direction of the effect. Based on this finding regarding decision making, I characterize introduction robust networks and Pareto efficient networks and show that they do not imply each other.

I now go through the reasoning behind the above results. The first step is to figure out how a new link between two players, say i and j, affects the matching of other players.

3.1 The Influence of a New Link

To examine the influence of a new link on the matching of players, consider the following questions. First, if there is a new link between i and j, will the two matchings μ^*_{g+ij} and μ^*_g be different? Second, if different, whose partner will be changed? Finally, for an affected player, will he enjoy an improvement or endure an impairment in his matching partner?

For the first question. I define a link between i and j as *useful* under g if i and j are matched in the unique stable matching μ^*_g. It is easy to see that μ^*_{g+ij} is different from μ^*_g if and only if the new link between i and j is useful under $g+ij$. The reasoning is simple as we can see that μ^*_{g+ij} and μ^*_g are different if the link between i and j is useful because the two players who cannot be matched in μ^*_g are now matched in μ^*_{g+ij}. We verify that μ^*_{g+ij} and μ^*_g are the same if the link is not useful by going through the algorithm used to derive μ^*_{g+ij} and observing that all steps lead to the same outcomes as those taken to derive μ^*_g.

Also, note that for a link between i and j to be useful under $g+ij$, the two players that are not matched under g get matched under $g+ij$. This requires that they prefer each other to their original match in μ^*_g, which means

$$a_i > a_{\mu^*_g(j)} \text{ if } \mu^*_g(j) \neq j, \text{ and}$$
$$a_j > a_{\mu^*_g(i)} \text{ if } \mu^*_g(i) \neq i.$$

This, I arrive at Lemma 2 that characterizes the scenario when a new link between i and j has an effect on the matching outcomes of players.

Lemma 2. *Consider a network $g \in G$ and two players $i, j \in N$ where $g_{ij} = 0$. There exists a player $m \in N$ such that $\mu^*_{g+ij}(m) \neq \mu^*_g(m)$ if and only if the link between i and j is useful under $g+ij$:*

$$a_i > a_{\mu^*_g(j)} \text{ if } \mu^*_g(j) \neq j, \text{ and}$$
$$a_j > a_{\mu^*_g(i)} \text{ if } \mu^*_g(i) \neq i.$$

I now discuss whose matching will be affected by a useful new link. Recall the algorithm specified in Lemma 2; when it is executed, a sequence of steps are taken and we can observe that given any network structure and ability endowment, the sequence is unique. Based on this unique sequence, I define player i as a *leading player* of player j under network g if i exits from the algorithm used to derive μ^*_g earlier than j. Lemma

3 shows that with a new link between i and j, leading players of both i and j under g will not be affected by the link.

Lemma 3. *Consider a network $g \in G$ and two players $i, j \in N$ where $g_{ij} = 0$. If player $m \in N$ is a leading player of both i and j under g, then $\mu^*_{g+ij}(m) = \mu^*_g(m)$.*

With the help of Lemmas 2 and 3, I arrive at the first result regarding introduction incentives of players: an introducer is indifferent towards whether to introduce a pair of players, if at least one of them is less capable than the introducer, because the introducer must be a leading player of the two if the introduction creates a useful link. This will be elaborated in the next subsection. For now, I continue investigating the influence of a new link.

So, we know leading players of both i and j under g will not be affected by a new link between i and j, but what about other players? I define an *alternating path* under network g as a path where pairs of players on the path are alternatively matched to each other and not matched to each other in μ^*_g. Figure 1 gives an example of an alternating path. I use a red line to denote a useful link and a black line to indicate a useless link. In this path, player 1 is matched with player 2, player 2 is not matched with player 3, player 3 is matched with player 4, and player 4 is not matched with player 5. I show that all the players that are affected by a new link between i and j sit on two alternating paths that start from i and j respectively. Formally, a path $\{i_0, i_1, ..., i_k\}$ in g is an alternating path if $\mu^*_g(i_0) = i_1$, $\mu^*_g(i_2) = i_3$, ..., $\mu^*_g(i_{k-1}) = i_k$ when k is odd and $\mu^*_g(i_{k-2}) = i_{k-1}$ when k is even.

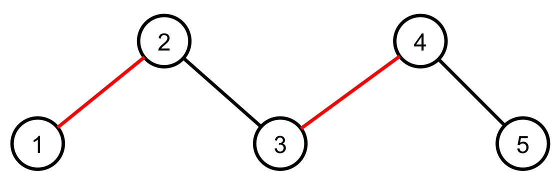


Figure 1: An example of an alternating path

Lemma 4. *Consider a network $g \in G$ and two players $i, j \in N$ where $g_{ij} = 0$. There exist two alternating paths $\{i = i_0, i_1, ..., i_k\}$ and $\{j = j_0, j_1, ..., j_{k'}\}$ in g such that for any player $m \in N$, $\mu^*_{g+ij}(m) \neq \mu^*_g(m)$ if and only if $m \in \{i_0, i_1, ..., i_k, j_0, j_1, ..., j_{k'}\}$.*

The intuition behind Lemma 4 is the following. If the matching of a player m is affected by a new link between i and j, ignoring the case where m is single under g or $g + ij$, m must have broken his original match under g, say with x, and formed a new match under $g + ij$, say with y, as shown in Figure 2. Since player x is originally matched with m under g and player y is not matched with m under g, we can already see that m sits on a short alternating path in g that consists of x, m, and y. Moreover, as x and y are affected by the new link between i and j as well, they are also part of an alternating path that contains themselves, m, and another player. By repeating this reasoning, we can extend the alternating path with affected players on it. This extension must end with a source of changes, which is the matching between i and j that is facilitated by their new link. So, players affected by a new link between i and j must sit on an alternating path that starts from i or j.

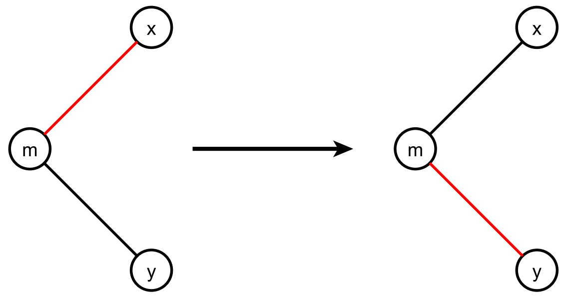
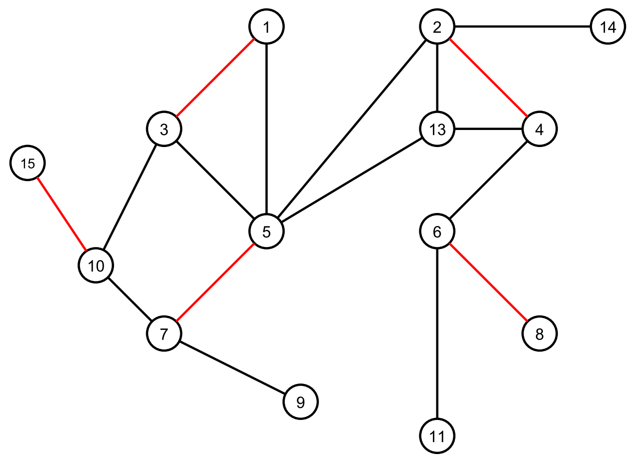


Figure 2: The new and original partners of m

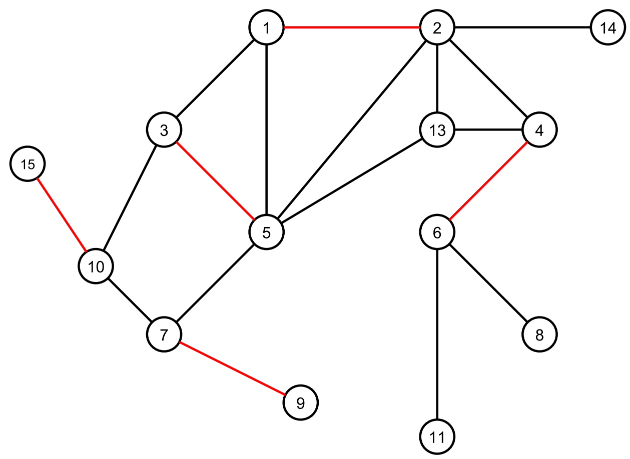
I illustrate the alternating path pattern of affected players with an example in Figure 3 where a label on a node represents the ability ranking of a player and red and black links represent useful and useless links respectively as in previous figures. Figure 3 (a) shows the unique stable matching under a network g, and Figure 3 (b) shows the new unique stable matching under the network with an additional link between players 1 and 2. Upon comparing Figures 3 (a) and 3 (b), we can easily find the set of players whose matchings are influenced by the new link, which is illustrated in Figure 3 (c) where the nodes of affected players are painted red. Also in Figure 3 (c), there are two red paths starting from player 1 and player 2 respectively that connect the affected players. The solid red links are useful under the original network g and the dashed red links are not useful under g. These two paths are the alternating paths mentioned in Lemma 4.

Finally, I illustrate the direction of influence a new link between i and j has on an affected player. This depends on the parity of the alternating path length from i or j to the affected player. An odd-length path leads to an improvement and an even-length path leads to an impairment. To see why, consider a player Alice who is originally matched to i under g and hence there is an alternating path from i to her that is of length one. As i is matched to j under $g + ij$, Alice needs to find a new partner. She is worse off, as she would have otherwise chosen her new partner Bob over i had the network structure not been changed. Then, Bob obtains a new partner and must be better off, as he would otherwise choose to stay with his original partner Clara under g. Note that he has an alternating path from i to him that is of length two. Continuing with this reasoning, we can see that Clara, who has an alternating path of length three from i to her, is worse off since she lost Bob. And the new partner of Clara, Dillen, who has an alternating path of length four from i to him, is better off with the new match to Clara. By induction, it can be shown that the parity of the alternating path from i or j to an affected player pins down the direction of effect to the player. This finding is summarised in Lemma 5.

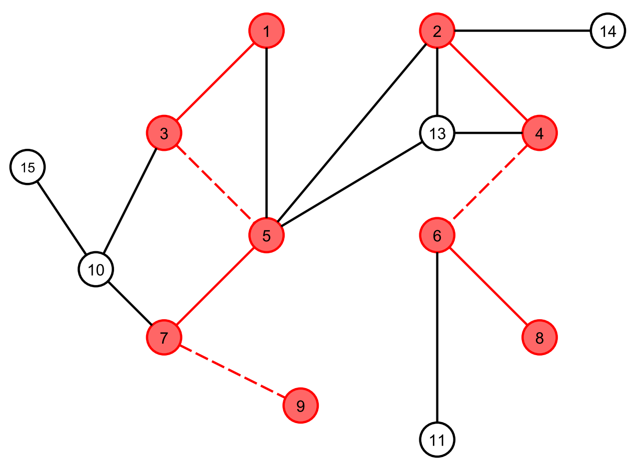
Lemma 5. *Consider a network $g \in G$ and two players $i, j \in N$ where $g_{ij} = 0$. There*



(a) The unique stable matching μ_g^*



(b) The unique stable matching μ_{g+12}^*



(c) Players affected by a new link between 1 and 2

Figure 3: Alternating paths and players affected by a new link

exist two alternating paths $\{i = i_0, i_1, ..., i_k\}$ and $\{j = j_0, j_1, ..., j_{k'}\}$ in g such that for any player $m \in N$, $\mu^*_{g+ij}(m) \succ_m \mu^*_g(m)$ if and only if $m \in \{i_0, i_2, ..., i_{2\lfloor\frac{k}{2}\rfloor}, j_0, j_2, ..., j_{2\lfloor\frac{k'}{2}\rfloor}\}$; and $\mu^*_g(m) \succ_m \mu^*_{g+ij}(m)$ if and only if $m \in \{i_1, i_3, ..., i_{2\lceil\frac{k}{2}\rceil-1}, j_1, j_3, ..., j_{2\lceil\frac{k'}{2}\rceil-1}\}$.

One last thing to note is that the abilities of players on the alternating path have a certain pattern. In Figure 3 (c), notice that the starting player on a path is more capable than the third player, the third player is more capable than the fifth player, and the second player on the path is more capable than the fourth player. There is a descending ability pattern every other player. The intuition behind this pattern is that for a positively affected player k who is the kth player on the path (so the length of the path from the introduced player to him is $k-1$, which is an even number), he wins a better partner who is the $(k-1)th$ player and abandon his original partner who is the $(k+1)th$ player on the alternating path, so $a_{k-1} > a_{k+1}$. Also, as the $(k-1)th$ player was not willing to partner with the kth player under g but chose to match with the $(k-2)th$ player, $a_{k-2} > a_k$. I now summarize all the findings regarding the influence of a new link on the matchings of players in Lemma 6.

Lemma 6. *Consider a network $g \in G$ and two players $i, j \in N$ where $g_{ij} = 0$. There exists a player $m \in N$ such that $\mu^*_{g+ij}(m) \neq \mu^*_g(m)$ if and only if the link between i and j is useful under $g + ij$:*

$$a_i > a_{\mu^*_g(j)} \text{ if } \mu^*_g(j) \neq j \text{ ,and}$$
$$a_j > a_{\mu^*_g(i)} \text{ if } \mu^*_g(i) \neq i.$$

There exist two alternating paths $\{i = i_0, i_1, ..., i_k\}$ and $\{j = j_0, j_1, ..., j_{k'}\}$ in g where $a_{i_{l+2}} > a_{i_l}$ for $l = 0, 1, ..., k-2$ and $a_{j_{l+2}} > a_{j_l}$ for $l = 0, 1, ..., k'-2$, such that for any player $m \in N$, $\mu^*_{g+ij}(m) \succ_m \mu^*_g(m)$ if and only if $m \in \{i_0, i_2, ..., i_{2\lfloor\frac{k}{2}\rfloor}, j_0, j_2, ..., j_{2\lfloor\frac{k'}{2}\rfloor}\}$; and $\mu^*_g(m) \succ_m \mu^*_{g+ij}(m)$ if and only if $m \in \{i_1, i_3, ..., i_{2\lceil\frac{k}{2}\rceil-1}, j_1, j_3, ..., j_{2\lceil\frac{k'}{2}\rceil-1}\}$.

3.2 When to Make an Introduction

I show two results regarding the incentives to make an introduction. First, if one of the neighbours a player introduces is less capable than him, then his matching would not be influenced and he will make such an introduction if he is generous and does nothing if he is not. Second, if a player is generous, a sufficient condition for him to introduce two neighbours more capable than him is that there is no alternating path with a descending ability pattern from any of the introduced players to him that is of an odd length. If a player is not generous, a necessary condition for him to introduce two neighbours more capable than him is that there is an alternating path with a descending ability pattern from one of the introduced players to him that is of an even length.

The first result is derived with the help of Lemmas 2 and 3. When there is a new link between i and j, it either does not facilitate a matching for i and j under $g + ij$ (the link is not useful under $g + ij$) so it does not influence the matching of any player, or it makes i and j match under $g + ij$. In the latter case, the introducer must have been matched to a player who has higher ability than both i and j and has exited from the matching algorithm earlier than i and j (the introducer is a leading player of both i

and j under g). If not, then the introducer and the more capable player between i and j would match as he is better than the weaker player between i and j, contradicting the matching between i and j under $g + ij$. In summary, an introducer does not face the risk of losing a partner he wants by making a less capable neighbour meet another neighbour because he will not be in a position of waiting to be selected by either of the neighbours if the introduction affects the matching of players. This result is formalized in Proposition 1.

Proposition 1. *For any player $m \in N$ and network $g \in G$, if $i, j \in N_m(g)$ and $a_m > \min\{a_i, a_j\}$, then $\mu^*_{g+ij}(m) = \mu^*_g(m)$ and m will choose to introduce i and j if he is generous and do nothing if he is not.*

The second result follows directly from Lemma 6. I simply translate the effect of introduction demonstrated in Lemma 6 into conditions for when players want to or do not want to facilitate an introduction.

Proposition 2. *For any player $m \in N$ and network $g \in G$, if $i, j \in N_m(g)$ and $a_m < \min\{a_i, a_j\}$, then*

(i) a generous player m will introduce i and j if there does not exist an alternating path $\{i_0, i_1, ..., i_k\}$ in g with $a_{i_{l+2}} > a_{i_l}$ for any $l \in \{0, 1, ..., k-2\}$ such that $i_0 \in \{i, j\}$ and $m \in \{i_1, i_3, ..., i_{2\lceil \frac{k}{2} \rceil - 1}\}$, and

(ii) a non-generous player m will introduce i and j only if there exists an alternating path $\{i_0, i_1, ..., i_k\}$ in g with $a_{i_{l+2}} > a_{i_l}$ for any $l \in \{0, 1, ..., k-2\}$ such that $i_0 \in \{i, j\}$ and $m \in \{i_2, i_4, ..., i_{2\lfloor \frac{k}{2} \rfloor}\}$.

This characterization based on alternating paths and their lengths has two implications. First, it shows that players can benefit from an introduction for other players. This is, to some extent, surprising. Since players in our model are competing for good partners, it seems that an introduction of two neighbours can only intensify the competitions a player faces and reduces his chance of having a good partner. Nonetheless, if the two neighbours are matched with the new link, they withdraw from competitions for other players who the introducer might want to partner with and the introducer can benefit from an introduction through this effect. Note also that it can take several steps for the influence to reach the introducer. An introducer can benefit from introducing Alice and Zelda because Alice frees Bob, who gets a new match with Clara, who frees Dillen, who is now willing to match with the introducer and is preferred by the introducer to his current partner.

Second, since affected players are alternatively better off and worse off given their positions on the two alternating paths, around half of them would have their matchings improved as a result of an introduction and half of them would have their matchings impaired. The chances of an introducer benefiting from an introduction are not low. This implies that even in an environment where quality neighbours are scarce resources and there is no premium gained from introducing two neighbours, introductions can still prevail. The results justify the common assumption in network formation literature that links can be formed via meeting friends of friends.

3.3 Introduction Robustness and Pareto Efficiency

With the understanding developed in the previous subsections, I now characterize conditions for a network to be introduction robust and to be Pareto efficient.

The conditions for a network to be introduction robust follow directly from Propositions 1 and 2.

Corollary 1. *When players are generous, if a network $g \in G$ is introduction robust, then for all $m \in N$ and $i, j \in N_m(g)$ with $g_{ij} = 0$, $a_m < \min\{a_i, a_j\}$, and there is an alternating path $\{i_0, i_1, ..., i_k\}$ in g with $a_{i_{l+2}} > a_{i_l}$ for any $l \in \{0, 1, ..., k-2\}$ such that $i_0 \in \{i, j\}$ and $m \in \{i_1, i_3, ..., i_{2\lceil \frac{k}{2} \rceil - 1}\}$.*

When players are not generous, a network $g \in G$ is introduction robust if for all $m \in N$, there does not exist $i, j \in N_m(g)$ with $g_{ij} = 0$ such that $a_m < \min\{a_i, a_j\}$ and there is an alternating path $\{i_0, i_1, ..., i_k\}$ in g with $a_{i_{l+2}} > a_{i_l}$ for any $l \in \{0, 1, ..., k-2\}$ such that $i_0 \in \{i, j\}$ and $m \in \{i_2, i_4, ..., i_{2\lfloor \frac{k}{2} \rfloor}\}$.

Regarding Pareto efficiency, I show that a network g is Pareto efficient if and only if at most one player is not matched in the unique stable matching under g.

Proposition 3. *A network $g \in G$ is Pareto efficient if and only if*

$$|\{m \in N : \mu_g^*(m) = m\}| \leq 1.$$

Here is the proof for this result. If there are two players i and j who are both unmatched in μ_g^*, then it must be that $g_{ij} = 0$. Also, all players that have a partner in μ_g^* are leading players of i and j in g. Adding a link between i and j would not affect the matching of other players (since they are leading players of i and j) and would facilitate a match between i and j and increase their utilities. When no more than one player remains single in μ_g^*, it is easy to see that no Pareto improvements can be made; improving the matching of one player must lead to the impairment of the matching of another player.

I now show that introduction robustness and Pareto efficiency do not imply each other.

Proposition 4. *Regardless of whether players are generous or not, there exists a network $g \in G$ that is introduction robust but not Pareto efficient and a network $g' \in G$ that is Pareto efficient but not introduction robust.*

This result can be shown with two examples. First, an empty network where $g_{ij} = 0$ for all $i, j \in N$ is introduction robust but not Pareto efficient. When players are not generous, we also show that a connected network can be introduction robust but not Pareto efficient. See Figure 4 for an example. There are four players in the network with labels indicating their ability ranks. The unique stable matching in the network features a single match between players 1 and 3. When players are not generous, this network is introduction robust since player 3 would not benefit from introducing players

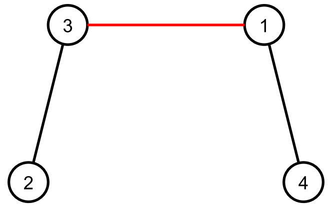
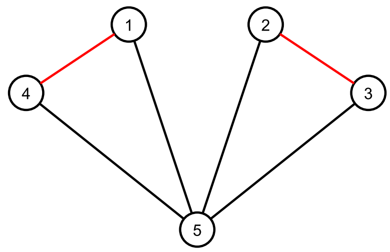
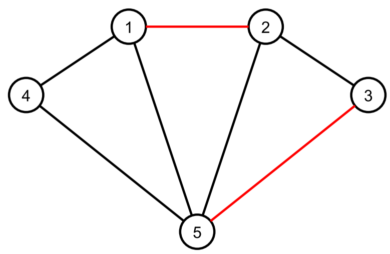


Figure 4: An introduction robust network that is not Pareto efficient



(a) The original network



(b) Player 5 benefits from an introduction

Figure 5: A Pareto efficient network that is not introduction robust

1 and 2, and player 1 would not benefit from introducing players 3 and 4. This network is not Pareto efficient since an additional link between players 3 and 4 would lead to a Pareto improvement.

Now I show that a Pareto efficient network is not necessarily introduction robust with the example depicted in Figure 5. In the five-player network in Figure 5 (a), the unique stable matching features a match between players 1 and 4 and a match between players 2 and 3. This network is Pareto efficient as only one player is not matched. However, it is not introduction robust since player 5 has an incentive to introduce player 1 and player 2. The network with the introduction is depicted in Figure 5 (b). We can see that now player 5 is matched with player 3 who is a better partner for him.

Thus, we have observed an absence of alignment between introduction robustness and Pareto efficiency.

4 Other Matching Contexts

In the baseline model, players have preference over networks that are determined by a one-sided one-to-one matching process. In this section, I investigate how players choose to make introductions when they are divided into two groups and either need to find a partner from the other group or want to collaborate multiple times with players from the other group.[9] I show that the intuitions derived under the baseline model remain robust for different matching contexts.

4.1 Two Sided One-to-One Matching

Consider a scenario where players are endowed with different abilities, situated in a network, and need to go through a one-to-one matching process as in the baseline model. They are now, however, divided into two groups and can only match with a neighbour in the other group. I analyze how players, aiming for a more attractive partner, decide whether to introduce two neighbours in this context.

The formal setup of the extension iss as follows. The specifications in Section 2.1 remain unchanged. For the matching process, players are divided into two groups M and W where $M \cup W = N$ and $M \cap W = \emptyset$. For ease in notation, let M be a set of men and W be a set of women. Given a network structure g, a feasible matching $\mu_g : N \to N$, other than satisfying the condition that $\mu_g(i) = j$ if only if $j \in N_i(g) \cup \{i\}$ and $\mu_g(j) = i$, must feature $\mu_g(m) \in \{m\} \cup W$ for all $m \in M$ and $\mu_g(w) \in \{w\} \cup W$ for all $w \in W$. The preference of players over partners is the same as in the baseline model. A matching μ_g in the two-sided one-to-one matching environment is stable under a network g if it is blocked by any player $i \in N$ (there does not exist $i \in N$ with $i \succ_i \mu_g(i)$) and it is not blocked by any pair of a man and a woman $(m, w) \in M \times W$ (there does not exist $(m, w) \in M \times W$ with $w \succ_m \mu_g(m)$, $m \succ_w \mu_g(w)$ and $w \in N_g(m)$).

With the additional restriction on matching, there is still a unique stable matching given any network $g \in G$, which can be obtained with the algorithm specified in Lemma 7.

Lemma 7. *For all $g \in G$, the two-sided one-to-one matching process admits a unique stable matching μ_g^* which can be derived with the following algorithm:*

(i) create a reduced network g^r where all links between players in the same group are deleted,

(ii) neglect the group membership of players and implement the matching algorithm specified in Lemma 1 to obtain $\mu_{g^r}^*$, and

(iii) for any player i, make $\mu_g^*(i) = \mu_{g^r}^*(i)$.

Given the change in the matching process, players have preferences for networks and hence may want to make introductions to change the network structure in their favour. I first show that the matching of a player, say w, stays unchanged if she introduces (i) a pair of same-sex neighbours and (ii) a woman who is less capable than

[9] The two-sided one-to-one matching and two-sided many-to-many matching environments are widely studied in the literature. For an overview, see Roth and Sotomayor (1992).

herself and any man or a man who is less capable than her current partner and any woman. An introduction of two same-gender neighbours is not consequential because the new link cannot be useful. The second situation where an introduction does not influence the introducer is analogous to Proposition 1 that we have for the baseline model with the only difference being in whose abilities we compare. Since a woman is competing with a woman and a man is competing with a man for a good partner, the ability comparison is only done within groups.

The matching of an introducer may change when introduces a pair of man and woman where the man is more capable than her current partner and the woman more capable than herself. As in the baseline model, whether a change takes place depends on if there is an alternating path with a descending ability pattern from one of the introduced players to her, and the direction of the change depends on the parity of the path length. Additionally, note that due to the two-sided matching requirement, neighbouring players on the alternating path must be of different genders and the length of the path must be even if it starts and ends at two players with the same gender and odd if it starts and ends at two players with different genders. Therefore, an alternating path that starts from the female neighbour can only make the introducer better off and an alternating path from the male neighbour can only make her worse off.

Men make decisions with symmetric reasoning regarding when to facilitate an introduction. The following lemma summarizes the results:

Lemma 8. *Consider the two-sided one-to-one matching environment: for any player $m \in N$, network $g \in G$, and players $i, j \in N_m(g)$ where $g_{ij} = 0$:*

(i) $\mu^*_{g+ij}(m) = \mu^*_g(m)$ if i and j belong to the same group, $a_m > a_l$, or $a_{\mu^*_g(m)} > a_{l'}$ where $l \in \{i, j\}$ is in the same group as m and $l' \in \{i, j\} \backslash \{l\}$,

(ii) $\mu^*_{g+ij}(m) \succ_m \mu^*_g(m)$ ($\mu^*_g(m) \succ_m \mu^*_{g+ij}(m)$) only if there is an altnertaing path $\{i_0, i_1, ..., i_k = m\}$ in g with $a_{i_{l+2}} > a_{i_l}$ for any $l \in \{0, 1, ..., k-2\}$ such that $i_0 \in \{i, j\}$ is in the same (opposite) group of m.

4.2 Two-Sided Many-to-Many Matching

I now examine another two-sided matching process where players are again divided into two groups, say managers M and assistants T, such that $M \cup T = N$ and $M \cap T = \emptyset$.[10] Each manager can hire multiple assistants to work on a project and each assistant can join multiple teams led by managers. For simplicity, I assume that all managers want to hire x assistants and that all assistants can join y teams. The heterogeneous capacity of managers and assistants would not qualitatively impact our results.

Players are endowed with different abilities and are situated in a network as in the baseline model. Given a network g, a matching μ_g is now a mapping from N to the set of all subsets of N and needs to satisfy

(i) $|\mu_g(m)| \leq x$ for all $m \in M$ and $|\mu_g(t)| \leq y$ for all $t \in T$,

(ii) $\mu_g(m) \subset N_m(g) \cap T$ for all $m \in M$ and $\mu_g(t) \subset N_t(g) \cap M$ for all $t \in T$, and

[10] The setup can also be employed to study the matching between upstream and downstream firms, between suppliers and buyers, and so on.

(iii) $t \in \mu_g(m)$ if and only if $m \in \mu_g(t)$ for all $m \in M$ and $t \in T$.

Players all wish to be matched with better partners and to have as many partners as possible before reaching capacity. Specifically, player i prefers a set of partners $N_1 = \{i_1, ..., i_{n_1}\}$ to $N_2 = \{j_1, ..., j_{n_2}\}$ where $a_{i_1} > a_{i_2} > ... > a_{i_{n_1}}$ and $a_{j_1} > a_{j_2} > ... > a_{j_{n_2}}$ if $n_1 \geq n_2$ and $a_{ik} \geq a_{jk}$ for all $k = 1, ..., n_2$ with at least one inequality being strict.

A matching μ_g in the two-sided many-to-many matching environment is stable under a network g if it is not blocked by any player $i \in N$: $\nexists i \in N, N' \subset \mu_g(i)$ such that

$$N' \succ_i \mu_g(i),$$

and it is not blocked by any pair of manager and assistant $(m, t) \in M \times T$: $\nexists (m, t) \in M \times T, N^1 \subset \mu_g(m), N^2 \subset \mu_g(t)$ such that

$$\begin{aligned} t \cup \mu_g(m) \backslash N^1 &\succ_m \mu_g(m), \\ m \cup \mu_g(w) \backslash N^2 &\succ_t \mu_g(t), \text{ and} \\ t &\in N_m(g). \end{aligned}$$

For this matching setup, there is a unique stable matching that can be derived with the algorithm in the following Lemma.

Lemma 9. *For all $g \in G$, the two-sided many-to-many matching process admits a unique stable matching μ_g^* which can be derived with the following algorithm:*

(i) create an extended network g^e where each manager $m \in M$ is replaced with x replicates $m_1, ..., m_x$ and each assistant $t \in T$ is repaced with y replicates $t_1, ..., t_y$, give a link to two replicated players i_a and j_b in g^e if and only if $g_{ij} = 1$,

(ii) implement the matching algorithm specified in Lemma 7 to obtain $\mu_{g^e}^*$ with the following additional step: delete the links between i_a and j_b for all a and b when a pair of players $i_{a'}$ and $j_{b'}$ is removed from the algorithm, and

(iii) make $\mu_g^*(m) = \{\mu_{g^e}^*(m_1), ..., \mu_{g^e}^*(m_x)\} \backslash \{m\}$ for all $m \in M$ and $\mu_g^*(t) = \{\mu_{g^e}^*(t_1), ..., \mu_{g^e}^*(t_y)\} \backslash \{t\}$ for all $t \in T$.

Given that the stable matching for the many-to-many matching environment is derived with the algorithm for the one-to-one matching environment, it is not surprising that the conditions for an introducer to benefit or suffer from an introduction in the one-to-one matching setup are also needed here. An introduction for a pair of neighbours from the same group as well as an introduction for a weaker neighbour does not influence the matching of the introducer. Here, a neighbour is weaker than the introducer if he has lower ability than the introducer when in the same group as the introducer and if he has lower ability than the partner of the introducer when in the opposite group of the introducer. The considerations an introducer has when choosing whether to introduce a manager and an assistant who are both stronger than him depends on if there is an alternating path with a descending ability pattern starting from an introduced neighbour in the introducer's own or opposite group to him. Proposition 5 summarizes the results for the two two-sided matching environments.

Proposition 5. *Under the two-sided one-to-one and two-sided many-to-many matching environments: for any player $m \in N$, network $g \in G$, and players $i, j \in N_m(g)$ where $g_{ij} = 0$:*

(i) $\mu^*_{g+ij}(m) = \mu^*_g(m)$ if i and j belong to the same group, $a_m > a_l$, or $a_{\mu^*_g(m)} > a_{l'}$ where $l \in \{i, j\}$ is in the same group as m and $l' \in \{i, j\} \backslash \{l\}$, and

(ii) $\mu^*_{g+ij}(m) \succ_m \mu^*_g(m)$ ($\mu^*_g(m) \succ_m \mu^*_{g+ij}(m)$) only if there is an altnertaing path $\{i_0, i_1, ..., i_k = m\}$ in g with $a_{i_{l+2}} > a_{i_l}$ for any $l \in \{0, 1, ..., k-2\}$ such that $i_0 \in \{i, j\}$ is in the same (opposite) group of m.

5 Introductions with Incomplete Information

In the previous sections, it is assumed that players have full knowledge of the network structure and ability of others. One could argue, however, that the complete information assumption is unrealistic and that finding and comparing alternating paths requires too much strategic reasoning from players that it is an almost impossible mission. To address this concern, this section examines introductions when players have limited information, or only use their information to a certain extent. I assume that players are situated in a randomly formed network where two players are (weakly) more likely to be linked if they have similar ability levels.[11] A player, other than knowing how the network is formed, has information about the identities and abilities of the two neighbours he considers introducing, his current partner, and himself. Players decide whether or not to make an introduction by comparing the probability of obtaining a better and a worse match from the introduction. There are three obvious results in this situation. First, the lack of information does not change the fact that an introduction involving a less capable neighbour would not affect the matching of the introducer. Second, an introducer should never introduce his current partner to a neighbour more capable than him. Third, an introducer should always introduce two more-capable neighbours when he does not have a partner at the status quo. When the choice a player faces does not fall into the scenarios just described, there is still a simple rule of thumb: he should make the introduction when he is more capable than his partner and not do so when he is less capable. The intuition for this result is that among all of the affected players, around half of them are better off and half of them are worse off from the introduction since the parity of the alternating path length determines the direction of the effect. The question then arises: who is more likely to be those benefited? The answer is the players that are more capable than their partners since they are under-matched and a turbulence caused by an introduction will have a mean preserving effect.

I now go through the model and results in detail.

[11] Many empirical studies, as summarized in McPherson et al. (2001), show that players with similar features are more likely to be linked.

5.1 Information and Preference

All assumptions in Sections 2.1 and 2.2 regarding the abilities of players and the matching process are kept. Players know the ability distribution a of the population but do not know much about the network structure. They believe that the network is randomly formed with the probability of players i and j being linked determined by their abilities: $prob(g_{ij} = 1|a_i, a_j) = p(a_i, a_j)$ where $p : \mathbb{R}_+^2 \to (0, 1)$ satisfies $p(x, y) = p(y, x) \geq p(x', y')$ if $|x - y| > |x' - y'|$. Other than this public information, each player knows who his partner is under the current network structure. The abilities of a player and his partner are also known to the player. When the choice of whether to introduce two players, i and j, occurs to a player m, player m knows (or takes into account) the identities of i and j (whether i or j is his current partner) and the abilities of i and j.

Note that by knowing who his current partner is and who he is considering to introduce, m also knows that he has links to $\mu_g^*(m)$, i, and j and that i and j are not linked. So, the information m has to solve a decision making problem is

$$I_m = \{a, p, m, \mu_g^*(m), i, j, g_{m\mu_g^*(m)} = g_{mi} = g_{mj} = 1, g_{ij} = 0\}$$

and he considers if he should link i and j.

I assume that an introduction is beneficial to m if he is more likely to obtain a better partner than a worse partner following the introduction:

$$prob[\mu_{g+ij}^*(m) \succ_m \mu_g^*(m)|I_m] \geq prob[\mu_g^*(m) \succ_m \mu_{g+ij}^*(m)|I_m].$$

The inequality is strict if the player is not generous.

5.2 Simple Decision-Making Rules Based on Abilities

I show that players can follow some simple rules to decide whether to facilitate an introduction given incomplete information.

To start with, as in the baseline model with complete information, if one introduces two neighbours, one of whom is less capable than him, then this would not affect his matching. Second, one should never introduce his current partner to a neighbour more capable than him. This is because even if he obtains a new partner with the introduction, the new partner who is freed by the other introduced neighbour can never be more capable than his current partner as otherwise, the other introduced neighbour would not switch to match with the current partner of the introducer. Third, he should always introduce two more-capable neighbours when he has no partner in the status quo. This is obvious since he can only be better off if he is affected, and he can be affected given that he is introducing two more capable neighbours. These three decision rules are summarized in Proposition 6.

Proposition 6. *In the incomplete information setting, for any player $m \in N$ considering whether to introduce two players $i, j \in N_m(g)$: (i) if $a_m > \min\{a_i, a_j\}$, then $\mu_{g+ij}^*(m) = \mu_g^*(m)$ and m would choose to make the introduction if he is generous and do nothing if not; (ii) if $a_m < \min\{a_i, a_j\}$ and $\mu_g^*(m) \in \{i, j\}$, then $prob[\mu_g^*(m) \succ_m$*

$\mu^*_{g+ij}(m)|I_m] > prob[\mu^*_{g+ij}(m) \succ_m \mu^*_g(m)|I_m] = 0$ and m would never make the introduction; and (iii) if $a_m < \min\{a_i, a_j\}$ and $\mu_{g^*}(m) = m$, then $prob[\mu^*_{g+ij}(m) \succ_m \mu^*_g(m)|I_m] > prob[\mu^*_g(m) \succ_m \mu^*_{g+ij}(m)|I_m] = 0$ and m would always make the introduction.

The reasoning underlying this proposition are all explained and discussed in the baseline model. The new intuition to obtain is for the situation when a non-single player considers whether to introduce two more-capable neighbours that are not his current partner, which is described in Proposition 7.

Proposition 7. *In the incomplete information setting, for any player $m \in N$ considering whether to make an introduction for two players $i, j \in N_m(g)$ where $a_m \min\{a_i, a_j\}$ and $\mu^*_g(m) \notin \{m, i, j\}$: $prob[\mu^*_{g+ij}(m) \succ_m \mu^*_g(m)|I_m] > prob[\mu^*_g(m) \succ_m \mu^*_{g+ij}(m)|I_m]$ if and only if $a_m > a_{\mu^*_g(m)}$, so player m makes the introduction if and only if m is more capable than his partner under network g.*

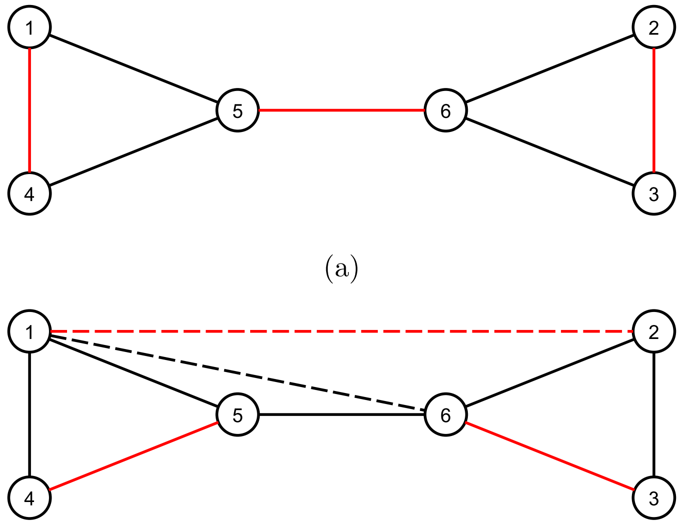
The intuition of the result is the following. Suppose a player m is affected by an introduction, then his partner must also be affected and the pair must experience different effects from the introduction: one of them will get a better partner, leaving the other abandoned for a worse match. The question then arises: which one of them is more likely to be better off? The answer is the more capable one since he is more likely to be linked to capable players and is more likely to be chosen by capable players. This result predicts that under matched players are more likely to act as introducers, which is a statement that can be tested empirically.

6 Conclusion

This paper studies introductions, which is a major way to connect people. In my model, links in the network serve as a contact book for players who wish to find a partner. Obviously, all players want to amplify their contact books so that their chances of getting a better partner increase. However, if links can only be formed via introductions, will a player do a favour for his neighbours by making the introduction?

I first show that a player's own matching will not be influenced when he makes an introduction where at least one of the two being introduced is weaker than him. I then characterize when a player will get a better match or worse match by introducing a pair of players both more capable than him. Given this, we understand when a (myopic) player wants to introduce a particular pair of players. Finally, I move to define a network as introduction robust when no (myopic) player wants to make an introduction.

Nonetheless, the introduction robustness notion defined in this chapter is short-sighted. We know that players choose between staying in the current network g and making an introduction by comparing their utilities before and after the introduction. In this paper, when I specify a player's utility after an introduction for i and j, I define it as the payoff from the unique stable matching under network $g + ij$. However, this is not sophisticated as a far-sighted player knows that following his introduction for i and j, the network changes and other players can make further introductions. It is wiser for the player to evaluate his utility after the introduction as the utility from a network with all introductions that will emerge after the introduction.



(a)

(b)

Figure 6: Far-sightedness and introductions

I illustrate this with an example in Figure 6. As in the network depicted in Figure 6 (a), when players are not generous, no one has an incentive to introduce. However, Figure 6 (b) shows that if player 5 makes an introduction for player 1 and player 6, then player 6 will want to introduce player 1 and player 2 as this enables him to match with player 3 instead of player 5. With this introduction made by player 6, player 5 is now matched with player 4 which makes him better off. Hence, with future possible introductions taken into account, player 5 will want to introduce player 1 and 6. This consideration may lead us to more sophisticated stability notions like far-sighted introduction robustness.

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A Proofs

Proof for Lemma 1

To begin with, I show μ_g^* derived from the algorithm is stable. We already know that μ_g^* will not be blocked by any single player since players prefer to be matched than to stay alone. So, I only need to show that μ_g^* is not blocked by any pairs of players. Suppose the pair $(i{,}j)$ blocks the matching μ_g^*, which requires

$$\begin{aligned} a_j &> a_{\mu_g^*(i)}, \\ a_i &> a_{\mu_g^*(j)}, \text{ and} \\ j &\in N_i(g). \end{aligned}$$

Without loss of generality, suppose $a_i > a_j$, in the algorithm that leads to μ_g^*, i may get picked by someone, picks someone, or stays alone. If i is picked up, then it must be $a_{\mu_g^*(i)} > a_i > a_j$. A contradiction. If i picks someone, since j is not picked by i, it indicates that either j is not as capable as i's match in μ_g^*: $a_{\mu_g^*(i)} > a_j$, or j is already picked by a more capable player: $a_{\mu_g^*(j)} > a_i$. A contradiction. Suppose i is left alone in μ_g^* with j being his neighbour, it must be that j is already picked by a more capable player: $a_{\mu_g^*(j)} > a_i$. A contradiction. So, μ_g^* can not be dominated by any pair of players. Therefore, μ_g^* is a stable matching.

Now, I prove that μ_g^* is the unique stable matching. Suppose there is another matching μ_g' that is stable, then define the set $N' = \{i \in N \mid \mu_g'(i) \neq \mu_g^*(i)\} \neq \emptyset$ as the set of players who get a different matching partner under μ_g' and μ_g^*. Let player j be the most capable players in this set. Also, suppose that

$$\begin{aligned} \mu_g^*(j) &= m, \\ \mu_g'(j) &= n, \text{ and} \\ \mu_g'(m) &= k. \end{aligned}$$

It is obvious that m, n, and k belong to N' as well. I show that the pair $(j,\ m)$ blocks the matching μ_g', hence proving that there are no other stable matching other than μ_g^*: Since j is the most capable player in N', it is obvious that $a_j > a_k$, which makes m want to deviate from his match under μ_g' to j. Also, it can be shown that $a_m > a_n$. Suppose that this is not the case, then in the matching algorithm, it must be that n has been picked by another player, say l, before it can be picked by j. Hence, $a_l > a_j$, and $l \in N'$ since player l is not matched with n at μ_g'. This is in contradiction with j being the most capable player in N'. Given $a_j > a_k$ and $a_m > a_n$, $(j{,}m)$ blocks the matching μ_g'.

Hence μ_g^* is the unique stable matching.

Proof for Lemma 2

Refer to the discussion preceding the Lemma.

Proof for Lemma 3

First, if a new link between i and j is not useful, then from Lemma 2, the matching for m is not changed. If the new link is useful, without loss of generality, assume $a_i > a_j$. I first show that i is a leading player of j under g. Suppose that this is not the case, then since $a_j < a_i$, for j to be the leading player instead, it must be that $a_{\mu_g^*(j)} > a_i$. But since the link between i and j is useful, we need that $a_i > a_{\mu_g^*(j)}$. A contradiction. So, i is a leading player of j under g. Now, apply the algorithm used to derive the unique stable matching in Lemma 1 to networks g and $g + ij$, the steps before i's matching will be identical for the two runs. Hence, the matchings of leading players of i under g, who are also leading players of j under g, are not influenced by the new link. Moreover, these players are also leading players of i and j under $g + ij$.

Proof for Lemma 4

If there is a useful new link between i and j, the set of affected players $A = \{m \in N : \mu_g^*(m) \neq \mu_{g+ij}^*(m)\}$ can be characterized with the following process. Define $A_{k+1} = A_k \cup \{m \in N : \mu_g^*(m) \in A_k\} \cup \{m \in N : \mu_{g+ij}^*(m) \in A_k\}$. This function specifies an augmentation process for a player set. Start with the set $A_0 = \{i, j\}$ and apply the above augmentation iteratively to the set A_0 and obtain $A_1, A_2, ...$ Since the number of players is finite, there will be a point where the augmentation stops with $A_{K+1} = A_K$. I show that $A_K = A$.

First, I prove $A_K \subset A$. Since the link between i and j is useful, $A_0 \subset A$. Moreover, observe that if $A_k \subset A$, then $A_{k+1} \subset A$, because if a player is affected, then his original and new partners are affected as well. By induction, $A_K \subset A$.

Then, I prove $A \subset A_K$. Suppose there exists $m_0 \in A$ such that $m_0 \notin A_K$. Since $m_0 \in A$, he is either better off or worse off. Suppose m_0 is better off, m_0 must have a new partner $m_1 = \mu_{g+ij}^*(m_0)$ that he prefers to his original match. There are two possible situations that make $m_1 \neq \mu_g^*(m_0)$ match to m_0 under $g + ij$. One, both m_0 and m_1 are better off because there is a new link between them. Two, m_1 is worse off because $m_2 = \mu_g^*(m_1)$ finds a better match under $g + ij$ and leaves m_1.

In the second situation, m_2 is better off and we can do the same induction to m_2 again. This reasoning will not stop unless m_{2k} is better off under the first situation. The reasoning cannot proceed endlessly because there is a finite set of players. So, it must be that there is a new link between m_{2k} and m_{2k+1} for some k, indicating m_{2k} and m_{2k+1} are i and j, respectively. But then, $m_{2k} \in A_0$, $m_{2k-1} \in A_1$, $m_{2k-2} \in A_2$, ..., hence, $m_0 \in A_{2k}$. And since $A_k \subset A_K$ for k=1, 2, ..., $m_0 \in A_K$, a contradiction.

For m_0 to be worse off, it can be proved similarly that $m_0 \in A_K$ and hence, a contradiction. So, $A \subset A_K$.

As $A_K \subset A$ and $A \subset A_K$, $A_K = A$.

Now, explore the pattern for the set of players generated by the above process. It is obvious that $A_1 \backslash A_0$ consists of the original partners of i and j, $A_2 \backslash A_1$ consists of the new partners of players in $A_1 \backslash A_0$, $A_3 \backslash A_2$ consists of the original partners of players in $A_2 \backslash A_1$, and so on. The set A_k develops itself is by alternatively tracing the original and new partners of players newly added to the set. Since the links between original partners are useful under g and the links between new partners are not useful under g, the process provides us with the set of affected players by drawing two alternating paths under g starting from i and j with a useful link. Hence, the set of affected players $A = \{m \in N : \mu_g^*(m) \neq \mu_{g+ij}^*(m)\}$ lie on two alternating paths under g, starting from i and j, respectively.

Proof for Lemma 5

To start with, notice that for two players who are matched under μ_{g+ij}^* but not μ_g^*, they cannot both obtain a better partner under μ_{g+ij}^* than under μ_g^* unless they are i and j, respectively, and for two players who are matched under μ_g^* but not μ_{g+ij}^*, they cannot both obtain a worse partner under μ_{g+ij}^* than under μ_g^*. For the former statement, if the players both obtain a better partner and they are not i and j, then they should choose to match under network g, contradicting μ_g^* being a stable matching. For the latter statement, if the two players both obtain a worse partner, then they would choose to stay with each other as in μ_g^*, contradicting μ_{g+ij}^* being a stable matching. This reasoning will be used in the proof for Lemma 5.

For a player m with $\mu_{g+ij}^*(m) \succ_m \mu_g^*(m)$, we know from Lemma 4 that m is on at least one of the alternating paths starting from i and j respectively that contain all of the affected players by a new link between i and j. Suppose the length of the path from i or j to m is odd (if m is on both paths, suppose the length of both paths is odd), since m is bettern off, m must have a partner under μ_{g+ij}^*. Let $m_0 = m$ and $m_1 = \mu_{g+ij}^*(m_0)$. If $m_1 \in \{i, j\}$, then $m_0 \in \{i, j\}$ as well, there is an alternating path from $m_0 = i$ or j to himself which is of distance 0, a contradiction. If $m_1 \notin \{i, j\}$, then m_1 is worse off given our observation at the start of the proof and the length of the alternating path(s) from i or j to m_1 must be even. Since m_1 is worse off, he must have an original partner $m_2 = \mu_g^*(m_1)$ who, according to the reasoning at the start of the proof, must be better off and the length of the alternating path(s) from i or j to m_2 must be odd. Reasoning inductively, observe that for $m_k = u_{g+ij}^*(m_{k-1})$ where k is an odd number, either (a) $m_k \in i, j$ so that m_k is better off which implies $m_{k-1} \in i, j$ and there is an alternating path from m_{k-1} to m_0 of length $k - 1$, an even number, a contradiction, or (b) m_k is worse off and the extension process continues. Since there is a finite number of players, there must exist an odd number K where $m_K \in i, j$. So there must be an alternating path from $m_{K-1} \in i, j$ to $m_0 = 0$ that is of even length. Note also that for all $k \in 0, 1, ..., K - 1$, the length of the alternating path from $m_{K-1} \in i, j$ to m_k is $K - 1 - k$. When k is even, so that $K - 1 - k$ is odd, m_k is worse off and when k is odd, so that $K - 1 - k$ is even, m_k is better off.

With the same logic, it can be shown that for any player m to be worse off under μ_{g+ij}^* than under μ_g^*, there must be an alternating path featuring affected players from i or j to m that is of odd length, and any player in between who has an odd path length from i or j to him is worse off and any player in between who has an even path length from i or j to him is better off.

Let i_k and $j_{k'}$ be two affected players that have the longest affected-players-formed alternating path length from i and j to them such that the parity of the path length matches the direction of matching change as described by Lemma 5, then all other affected players must lie between i and i_k or j and $j_{k'}$. With our previous reasoning, Lemma 5 is proved.

Proof for Lemma 6

Lemma 6 summarizes results from Lemmas 1 to 5. The only new thing to prove is that the two alternating paths specified in Lemma 5 satisfy the descending ability pattern. This can be found in the discussion preceding the Lemma.

Proof for Proposition 1

Suppose $a_i > a_j$, since at least one of i and j is less capable than m, $a_j < a_m$. Also, i, j, and m are connected to each other under $g + ij$.

If the link between i and j is not useful under network $g + ij$, then obviously $\mu_g^*(m) = \mu_{g+ij}^*(m)$. If the link is useful, since $a_j < a_m$, it indicates that i will prefer m to j, but i and j are matched, indicating that m has exited from the algorithm earlier than i. Hence, m is a leading player of i under $g + ij$. From the proof for Lemma 3, we know that m is a leading player of both i and j under g and $\mu_g^*(m) = \mu_{g+ij}^*(m)$.

Proof for Proposition 2

Proposition 2 follows directly from Lemma 6.

Proof for Corollary 1

Corollary 1 follows directly from Propositions 1 and 2.

Proof for Proposition 3

See the discussion succeeding the Proposition.

Proof for Proposition 4

See the discussion succeeding the Proposition.

Proof for Lemma 7

We know that the algorithm in Lemma 1 gives a unique stable matching for the one-sided one-to-one matching process, so to prove Lemma 7, we only need to show that a matching μ_g is stable if and only if it can be derived by $\mu_g(i) = \mu_{g^r}^*(i)$ for any $i \in N$ and $\mu_{g^r}^*$ is stable under the one-sided one-to-one matching environment.

I first show that μ_g is stable if $\mu_g(i) = \mu^*_{g^r}(i)$ for any $i \in N$ and $\mu^*_{g^r}$ is stable. Since $\mu^*_{g^r}$ is stable, there does not exist a pair $(m, w) \in M \times W$ such that

$$\begin{aligned} &w \succ_m \mu_{g^r}(m), \\ &m \succ_w \mu_{g^r}(w), \text{ and} \\ &w \in N_m(g^r). \end{aligned}$$

Since $\mu_g(i) = \mu^*_{g^r}(i)$ for any $i \in N$, and $g_{mw} = g^r_{mw}$ for any $(m, w) \in M \times W$, there does not exist a pair $(m, w) \in M \times W$ such that

$$\begin{aligned} &w \succ_m \mu_g(m), \\ &m \succ_w \mu_g(w), \text{ and} \\ &w \in N_m(g). \end{aligned}$$

This means that μ_g is stable because the matching μ_g cannot be blocked by a single player by definition and we have now shown μ_g cannot be blocked by any male and female pair.

I then show that μ_g is stable only if it can be derived by letting $\mu_g(i) = \mu^*_{g^r}(i)$ for any $i \in N$ and $\mu^*_{g^r}$ is stable. Suppose that this is not the case, then there exists a network g such that μ_g is stable in the two-sided one-to-one matching environment, but a matching μ_{g^r} under the reduced network of g defined by making $\mu_{g^r}(i) = \mu_g(i)$ is not stable in the one-sided one-to-one matching environment. Then, there must exist a pair $(m, w) \in M \times W$ such that

$$\begin{aligned} &w \succ_m \mu_{g^r}(m), \\ &m \succ_w \mu_{g^r}(w), \text{ and} \\ &w \in N_m(g^r). \end{aligned}$$

Since $\mu_g(i) = \mu^*_{g^r}(i)$ for any $i \in N$, and $g_{mw} = g^r_{mw}$ for any $(m, w) \in M \times W$, there must exist a pair $(m, w) \in M \times W$ such that

$$\begin{aligned} &w \succ_m \mu_g(m), \\ &gm \succ_w \mu_g(w), \text{ and} \\ &w \in N_m(g). \end{aligned}$$

This contradicts the statement that μ_g is stable in the two-sided one-to-one matching environment.

Proof for Lemma 8

In the proof for Lemma 7, it is shown that a matching μ_g is stable if and only if it can be derived by letting $\mu_g(i) = \mu^*_{g^r}(i)$ for any $i \in N$ and $\mu^*_{g^r}$ is stable in the one-sided one-to-one matching environment. Given this equivalence, the difference between μ^*_{g+ij} and μ^*_g in the two-sided matching environment is the same as the difference between

$\mu^*_{g+ij^r}$ and $\mu^*_{g^r}$ in the one-sided matching environment. Therefore, we can directly employ Lemmas 1 to 6 for results on the influence of a new link on the matching of players in the two-sided matching environment.

For the first statement in (i), since a new link between two players i and j in the same group is deleted in the reduced network, $g + ij^r = g^r$, $\mu^*_{g+ij}(m) = \mu^*_{g+ij^r}(m) = \mu^*_{g^r}(m) = \mu^*_g(m)$ for all $m \in M$. For the second statement in (i), suppose i and j are in different groups and the new link between them is useful in $g + ij^r$: if $a_m > a_l$ where $l \in \{i, j\}$ is in the same group as m, then m must be a leading player of i and j under g^r, indicating that the matching of m is not changed; if $a_{\mu^*_g(m)} > a_{l'}$ where $l' \in \{i, j\}$ is in the opposite group of m, then there cannot be an alternating path with a descending ability pattern that starts from either i or j and passes through $\mu^*_g(m)$ and hence, m. So, the matching of m is not changed.

For (ii), we know from Lemma 6 that for the matching of a player to be influenced in a positive way, there must be an alternating path with a descending ability pattern from an introduced player to him that is of even length in the reduced network g^r, which implies that there must also be such a path in the original network g. Given that all matches are between players from different groups, an even length alternating path must start and end at two players from the same group. Therefore, $\mu^*_{g+ij}(m) \succ_m \mu^*_g(m)$ only if there is an altnertaing path $\{i_0, i_1, ..., i_k = m\}$ in g with $a_{i_{l+2}} > a_{i_l}$ for any $l \in \{0, 1, ..., k-2\}$ such that $i_0 \in \{i, j\}$ is in the same group as m. Similarily, we can prove $\mu^*_g(m) \succ_m \mu^*_{g+ij}(m)$ only if there is an altnertaing path $\{i_0, i_1, ..., i_k = m\}$ in g with $a_{i_{l+2}} > a_{i_l}$ for any $l \in \{0, 1, ..., k-2\}$ such that $i_0 \in \{i, j\}$ is in the group opposite to that of m.

Proof for Lemma 9

We know that the algorithm in Lemma 7 gives a unique stable matching for the two-sided one-to-one matching process, so to prove Lemma 7, we only need to show that a matching μ_g is stable if and only if it can be derived by having $\mu^*_g(m) = \{\mu^*_{g^e}(m_1), ..., \mu^*_{g^e}(m_x)\} \backslash \{m\}$ for all $m \in M$ and $\mu^*_g(t) = \{\mu^*_{g^e}(t_1), ..., \mu^*_{g^e}(t_y)\} \backslash \{t\}$ for all $t \in T$, and $\mu^*_{g^e}$ is stable under the two-sided one-to-one matching environment.

For the 'if' part, since $\mu^*_{g^e}$ is stable, there does not exist a pair (m_a, t_b) such that

$$\begin{aligned} t_b &\succ_{m_a} \mu^*_{g^e}(m_a), \\ m_a &\succ_{t_b} \mu^*_{g^e}(t_b), \text{ and} \\ t_b &\in N_{m_a}(g^e). \end{aligned}$$

Since $\mu^*_g(m) = \{\mu^*_{g^e}(m_1), ..., \mu^*_{g^e}(m_x)\} \backslash \{m\}$ for all $m \in M$ and $\mu^*_g(t) = \{\mu^*_{g^e}(t_1), ..., \mu^*_{g^e}(t_y)\} \backslash \{t\}$ for all $t \in T$, and given the preference specification of the many-to-many matching process, there does not exist pairs (m, t) and (m_a, t_b) such that

$$\begin{aligned} \{t_b\} \cup \mu_g(m) \backslash \{\mu^*_{g^e}(m_a)\} &\succ_m \mu_g(m), \\ \{m_a\} \cup \mu_g(t) \backslash \{\mu^*_{g^e}(t_b)\} &\succ_t \mu_g(t), \text{ and} \\ t &\in N_m(g). \end{aligned}$$

This implies that there does not exist $(m,t) \in M \times T$, $N^1 \in \mu_g(m)$, and $N^2 \in \mu_g(t)$ such that

$$\{t\} \cup \mu_g(m) \backslash N^1 \succ_m \mu_g(m),$$
$$\{m\} \cup \mu_g(t) \backslash N^2 \succ_t \mu_g(t), \text{ and}$$
$$t \in N_m(g),$$

since players always prefer to have extra partners before reaching their matching capacity. For the same reason, condition (i) of a stable matching under the two-sided many-to-many matching environment is automatically satisfied for μ_g. So, μ_g is stable.

For the 'only if' part, suppose there exists a network g such that μ_g is stable in the two-sided many-to-many matching environment but the matching $\mu^*_{g^e}$ under the extended network of g is not stable in the two-sided one-to-one matching environment. Then there must exist pairs (m,w) and (m_a,t_b) such that

$$t_b \succ_{m_a} \mu^*_{g^e}(m_a),$$
$$m_a \succ_{t_b} \mu^*_{g^e}(t_b), \text{ and}$$
$$t_b \in N_{m_a}(g^e).$$

Since $\mu^*_g(m) = \{\mu^*_{g^e}(m_1), ..., \mu^*_{g^e}(m_x)\} \backslash \{m\}$ for all $m \in M$ and $\mu^*_g(t) = \{\mu^*_{g^e}(t_1), ..., \mu^*_{g^e}(t_y)\} \backslash \{t\}$ for all $t \in T$, there exists pairs (m,t) and (m_a,t_b) such that

$$\{t\} \cup \mu_g(m) \backslash \{\mu^*_{g^e}(m_a)\} \succ_m \mu_g(m),$$
$$\{m\} \cup \mu_g(t) \backslash \{\mu^*_{g^e}(t_b)\} \succ_t \mu_g(t), \text{ and}$$
$$t \in N_m(g),$$

contradicting μ_g being stable in the two-sided many-to-many matching environment.

Proof for Proposition 5

In the proof for Lemma 9, it is shown that a matching μ_g is stable if and only if it can be derived by having $\mu^*_g(m) = \{\mu^*_{g^e}(m_1), ..., \mu^*_{g^e}(m_x)\} \backslash \{m\}$ for all $m \in M$ and $\mu^*_g(t) = \{\mu^*_{g^e}(t_1), ..., \mu^*_{g^e}(t_y)\} \backslash \{t\}$ for all $t \in T$, and $\mu^*_{g^e}$ is stable under the two-sided one-to-one matching environment. Given this equivalence, the difference between μ^*_{g+ij} and μ^*_g in the many-to-many matching environment corresponds to the difference between $\mu^*_{g+ij^e}$ and $\mu^*_{g^e}$ in the one-to-one matching environment.

Now, for the first statement in (i), if i and j belong to the same group in g, then i_a and j_b belong to the same group in g^e for any a and b, so the matching of m_c is not changed for any c according to Lemma 8, which implies $\mu^*_{g+ij}(m) = \mu^*_g(m)$.

For the second statement in (i), if $a_m > a_l$ or $a_{\mu^*_g(m)} > a_{l'}$ where $l \in \{i,j\}$ is in the same group as m and $l' \in \{i,j\} \backslash \{l\}$, then $a_{mc} > a_{la}$ and $a_{\mu^*_g(m_c)} > a_{l'_b}$ for any a, b, and c, so the matching of m_c is not changed for any c according to Lemma 8, which implies $\mu^*_{g+ij}(m) = \mu^*_g(m)$.

For (ii), we know the matching of m is influenced positively or negatively by a new link between i and j only if there exists a number c where the matching of m_c is influenced positively or negatively by a set of new links between i_a and j_b for all a and b in network g^e. For this to be the case, there must be an alternating path featuring affected players from i_a or j_b for some a and b to m_c where the starting point of the path is in the same or opposite group as m_c. This implies that there is an alternating path featuring affected players from i or j to m where the starting point of the path is in the same or opposite group of m.

Proof for Proposition 6

For (i), the proof follows directly from Proposition 1.

For (ii), when $a_m < \min\{a_i, a_j\}$, Lemma 6 tells us that $prob[\mu^*_g(m) \succ_m \mu^*_{g+ij}(m)|I_m] > 0$ since $\mu^*_g(m) \neq m$. I now show that when $\mu^*_g(m) \in \{i, j\}$, $prob[\mu^*_{g+ij}(m) \succ_m \mu^*_g(m)|I_m] = 0$. Without loss of generality, let $\mu^*_g(m) = i$. First, I show there does not exist an alternating path $\{i = i_0, i_1, ..., i_k\}$ in g such that $m \in \{i_0, i_2, ..., i_{2\lfloor \frac{k}{2} \rfloor}\}$. This is obvious since the second player on an alternating path starting from i must be m. Now consider an alternating path with a descending ability patter from j: $\{j = j_0, j_1, ..., j_{k'}\}$. Suppose the new link between i and j is useful under $g + ij$, then $a_{\mu^*_g(m)} > a_{\mu^*_g(j)}$ where $\mu^*_g(j) = j_1$. Given the descending ability patter, $a_{\mu^*_g(m)} > a_{j_1} > a_{j_3} > ....$ Even if $m \in \{j_2, ..., j_{2\lfloor \frac{k}{2} \rfloor}\}$, $\mu^*_{g+ij}(m) \in \{j_1, j_3, ...\}$ and cannot be better than $\mu^*_g(m)$.

For (iii), since $\mu_{g^*}(m) = m$, $prob[\mu^*_g(m) \succ_m \mu^*_{g+ij}(m)|I_m] = 0$. If $a_m < \min\{a_i, a_j\}$, then from Lemma 6, $prob[\mu^*_{g+ij}(m) \neq \mu^*_g(m)|I_m] > 0$ so $prob[\mu^*_{g+ij}(m) \succ_m \mu^*_g(m)|I_m] > 0$.

Proof for Proposition 7

When player m is affected by an introduction for i and j, his partner under g is also affected and there must be an integer $l \in \mathbb{Z}^+$ where m and $\mu^*_g(m)$ are the two players i_{2l} and i_{2l+1} or j_{2l} and j_{2l+1} that are on the alternating path from i or j specified in Lemma 6.

Without loss of generality, consider $m, \mu^*_g(m) \in \{i_{2l}, i_{2l+1}\}$ for some l. It cannot be that i_{2l-1} has no links to m and $\mu^*_g(m)$. If i_{2l-1} has links to both m and $\mu^*_g(m)$, then $m = i_{2l}$ so that he is better off with the introduction if and only if $a_m > a_{\mu^*_g(m)}$ as player i_{2l-1} will select the more capable player between m and $\mu^*_g(m)$.

If i_{2l-1} has only one link to m and $\mu^*_g(m)$, then the one with the link to i_{2l-1} is player i_{2l} and is better off with the introduction. Suppose $a_{i_{2l-1}} > \max\{a_m, a_{\mu^*_g(m)}\}$, then given that players with similar abilities are more likely to have a link between them, $prob[m = i_{2l}] > prob[\mu^*_g(m) = i_{2l}]$ if and only if $a_m > a_{\mu^*_g(m)}$. Suppose $a_{i_{2l-1}} < \max\{a_m, a_{\mu^*_g(m)}\}$, since $a_{i_{2l-1}} > a_{i_{2l+1}}$, then $a_{i_{2l}} > a_{i_{2l-1}} > a_{i_{2l+1}}$, $m = i_{2l}$ if and only if $a_m > a_{\mu^*_g(m)}$.

Therefore, for any player $m \in N$ considering whether to introduce two players $i, j \in N_m(g)$ where $a_m < \min\{a_i, a_j\}$ and $\mu^*_g(m) \notin \{m, i, j\}$, $prob[\mu^*_{g+ij}(m) \succ_m \mu^*_g(m)|I_m] > prob[\mu^*_g(m) \succ_m \mu^*_{g+ij}(m)|I_m]$ if and only if $a_m > a_{\mu^*_g(m)}$.